University of Badji Mokhtar-Annaba **Faculty of Engineering Sciences** **Department of Science and Technology,** **2nd Year** **<u>Pr. L. BECHIRI</u>** **Module: Physics 3**		**2025-2026** **Tutorial 3 Solutions**

<u>Exercise 1:</u>

1)

a) In a mode of pulsation ω : $x(t) = A cos(\omega t + \varphi)$ *and* $\theta(t) = B cos(\omega t + \varphi)$
Substituting into the homogeneous system of equations.

$$\ddot{x} + 10x - \theta = 0$$

$$\ddot{\theta} + 10\theta - x = 0$$

We obtain

$$(10 - \omega^2)A - B = 0 \qquad (1)$$

$$-A + (10 - \omega^2)B = 0 \qquad (2)$$

$$\begin{vmatrix} (10 - \omega^2) & -1 \\ -1 & (10 - \omega^2) \end{vmatrix} \cdot \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

The values of ω are the roots of the equation :

$$\mathfrak{D} = \begin{vmatrix} 10 - \omega^2 & -1 \\ -1 & 10 - \omega^2 \end{vmatrix} = (10 - \omega^2)^2 - 1 = (11 - \omega^2)(9 - \omega^2) = 0$$

$$\Rightarrow \omega_1 = \sqrt{11}\frac{rad}{s} \; et \quad \omega_2 = \sqrt{9} = 3\,\frac{rad}{s} \quad \textit{The natural frequencies}$$

$$x_h = A_1 \cos(\omega_1 t + \varphi_1) + A_2 \cos(\omega_2 t + \varphi_2)$$

$$\theta_h = B_1 \cos(\omega_1 t + \varphi_1) + B_2 \cos(\omega_2 t + \varphi_2)$$

b) $F(t) = F_0\, cos\omega_{ex} t \Longrightarrow \; x_p(t) = a cos(\omega_{ex} t + \alpha_1)\; et\; \theta(t) = b cos(\omega_{ex} t + \alpha_2)$

Let's find a, b α_1 *et* α_2, and using the complex method.

$$\bar{x}(t) = \bar{a}e^{j\omega_{ex}t} = ae^{j\alpha_1}e^{j\omega_{ex}t} \qquad \bar{a} = ae^{j\alpha_1}$$

$$\bar{\theta}(t) = \bar{b}e^{j\omega_{ex}t} = ae^{j\alpha_2}e^{j\omega_{ex}t} \qquad \bar{b} = be^{j\alpha_2}$$

$$F(t) = \mathcal{R}(F_0\, e^{j\omega_{ex}t})$$

By substituting into the system of equations, we obtain :

$$(10 - \omega_{ex}^2)\bar{a} \qquad - \bar{b} = F_0$$

$$-\bar{a} \qquad + \quad (10 - \omega_{ex}^2)\bar{b} = 0$$

$\bar{a}$ and $\bar{b}$ to be determined using Cramer's rule. We have a system of two equations with two unknowns:

The principal determinant $\mathfrak{D}_p$:

$$\mathfrak{D}_p = \begin{vmatrix} (10-\omega_{ex}^2) & -1 \\ -1 & (10-\omega_{ex}^2) \end{vmatrix}$$

$$\bar{a} = \frac{\begin{vmatrix} F_0 & -1 \\ 0 & (10-\omega_{ex}^2) \end{vmatrix}}{\mathfrak{D}_p} = \frac{(10-\omega_{ex}^2)F_0}{\mathfrak{D}_p} = \frac{(10-\omega_{ex}^2)F_0}{(10-\omega_{ex}^2)^2-1}$$

$$\bar{b} = \frac{\begin{vmatrix} (10-\omega_{ex}^2) & F_0 \\ -1 & 0 \end{vmatrix}}{\mathfrak{D}_p} = \frac{F_0}{\mathfrak{D}_p} = \frac{F_0}{(10-\omega_{ex}^2)^2-1}$$

$$a = |\bar{a}|$$

$$b = |\bar{b}|$$

$$\alpha_1 = \alpha_2 = 0$$

$$ae^{j\alpha_1} = a(\cos\alpha_1 + jsin\alpha_1) = a(\arg(z_1) - \arg(z_2))$$

The forced oscillations can be expressed as:

$$x_p(t) = acos(\omega_{ex}t + \alpha_1) \;\; avec \;\; a = |\bar{a}| \;\; et \; \alpha_1 = arg\bar{a}$$

$$\theta_p(t) = bcos(\omega_{ex}t + \alpha_2) \;\; avec \;\; b = |\bar{b}| \;\; et \; \alpha_2 = arg\bar{b}$$

When a and b are real, the forced oscillations can be expressed as:

$$x_p(t) = acos(\omega_{ex}t) = \frac{(10-\omega_{ex}^2)F_0}{(10-\omega_{ex}^2)^2-1}cos(\omega_{ex}t)$$

$$\theta_p(t) = bcos(\omega_{ex}t) = \frac{F_0}{(10-\omega_{ex}^2)^2-1}cos(\omega_{ex}t)$$

c) The general expressions for $x_g(t)$ et $\theta_g(t)$:

$$x_g(t) = x_h + x_p = A_1\cos(\omega_1 t + \varphi_1) + A_2\cos(\omega_2 t + \varphi_2) + acos(\omega_{ex}t)$$

$$\theta_g(t) = \theta_h + \theta_p = B_1\cos(\omega_1 t + \varphi_1) + B_2\cos(\omega_2 t + \varphi_2) + bcos(\omega_{ex}t)$$

From equation (1), we express B in terms of A : $(10-\omega^2)A - B = 0$ on $B = (10-\omega^2)A$

$$B_1 = (10-\omega_1^2)A_1 = (10-11)A_1 = -A_1$$

$$B_2 = (10-\omega_2^2)A_2 = (10-9)A_2 = A_2$$

$$\begin{cases} x_g(t) = A_1 \cos(\sqrt{11}\ t + \varphi_1) + A_2 \cos(3\ t + \varphi_2) + \dfrac{(10-\omega_{ex}^2)F_0}{(10-\omega_{ex}^2)^2 - 1} cos(\omega_{ex} t) \\ \theta_g(t) = -A_1 \cos(\sqrt{11}\ t + \varphi_1) + A_2 \cos(3\ t + \varphi_2) + \dfrac{F_0}{(10-\omega_{ex}^2)^2 - 1} cos(\omega_{ex} t) \end{cases}$$

Exercise 2:

First method:

The torques' moments $-C\theta_1\overrightarrow{K}\ \ and\ -C\theta_2\overrightarrow{K}$, where $\overrightarrow{K}$ is the unit vector directed upward.

$$\vec{F}_1 = -ka(\theta_1 - \theta_2)\vec{\imath}$$

$$\vec{F}_2 = -ka(\theta_2 - \theta_1)\vec{\imath}$$

where the angles $\begin{cases} \theta_1 \\ \theta_2 \end{cases}$ are very small.

$$J\ddot{\theta}\overrightarrow{K} = Sum\ of\ the\ moments\ of\ forces\ with\ respect\ to\ the\ rotation\ axis = \sum \vec{M}(\vec{F}_{ex}).\overrightarrow{K}$$

$$\vec{M}(\vec{F}_1) = \overrightarrow{O_1A} \wedge \vec{F}_1 = 0_1A.ka(\theta_1 - \theta_2)\sin(\frac{\pi}{2} - \theta_1)\overrightarrow{K}$$

$$\vec{M}(\vec{F}_2) = \overrightarrow{O_2B} \wedge \vec{F}_1 = 0_2B.ka(\theta_2 - \theta_1)\sin(\frac{\pi}{2} - \theta_2)\overrightarrow{K}$$

$$\sin\left(\frac{\pi}{2} - \theta_1\right) = cos\theta_1 \approx 1$$

$$\sin\left(\frac{\pi}{2} - \theta_2\right) = cos\theta_2 \approx 1$$

$$\begin{cases} J\ddot{\theta}_1 = -C\theta_1 - ka^2(\theta_1 - \theta_2) = -(C + ka^2)\theta_1 + ka^2\theta_2 \\ J\ddot{\theta}_2 = -C\theta_2 - ka^2(\theta_2 - \theta_1) = -(C + ka^2)\theta_2 + ka^2\theta_1 \end{cases}$$

$$k = 5,\quad C = 90,\qquad J = 1\quad et\ \ a = 2$$

The equations of motion are written as

$$\begin{cases} \ddot{\theta}_1 + 110\theta_1 - 20\theta_2 = 0 \\ \ddot{\theta}_2 + 110\theta_2 - 20\theta_1 = 0 \end{cases}$$

In a mode with pulsation ω, the solutions take the form:

$$\theta_1 = Acos(\omega t + \varphi)$$

$$\theta_2 = Bcos(\omega t + \varphi)$$

$$\begin{cases} (110 - \omega^2)A - 20B = 0 \\ -20A + (110 - \omega^2)B = 0 \end{cases}$$

The natural frequencies are the roots of the equation:

$$\begin{vmatrix} 110 - \omega^2 & -20 \\ -20 & 110 - \omega^2 \end{vmatrix} = (110 - \omega^2)^2 - 20^2 = 0$$

$$(110 - \omega^2 + 20)(110 - \omega^2 - 20) = 0$$

$$(130 - \omega^2)(90 - \omega^2) = 0$$

$$\omega_1 = \sqrt{130} = 11{,}402\ rad/s \quad \text{et} \quad \omega_2 = \sqrt{90} = 9{,}487 rad/s$$

Second method:

$$\mathcal{L} = \mathrm{T} - \mathrm{V}$$

$$T = \frac{1}{2} J \dot{\theta}_1^{\,2} + \frac{1}{2} J \dot{\theta}_2^{\,2}$$

$$V = \frac{1}{2} C\theta_1^2 + \frac{1}{2} C\theta_2^2 + \frac{1}{2} k(a\theta_1 - a\theta_2)^2$$

with $C = k_t$, the torsional constant

$$\mathcal{L} = \frac{1}{2} J \dot{\theta}_1^{\,2} + \frac{1}{2} J \dot{\theta}_2^{\,2} - \frac{1}{2} C\theta_1^2 - \frac{1}{2} C\theta_2^2 - \frac{1}{2} k(a\theta_1 - a\theta_2)^2$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = 0 \qquad ①$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = 0 \qquad ②$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = 0 \qquad J\ddot{\theta}_1 + (C + ka^2)\theta_1 - ka^2\,\theta_2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = 0 \qquad J\ddot{\theta}_2 + (C + ka^2)\theta_2 - ka^2\,\theta_1 = 0$$

In a mode with pulsation ω, the solutions take the form:

$$\theta_1 = A\cos(\omega t + \varphi)$$

$$\theta_2 = B\cos(\omega t + \varphi)$$

$$\begin{cases} (110 - \omega^2)A - 20B = 0 \\ -20A + (110 - \omega^2)B = 0 \end{cases}$$

The natural frequencies are the roots of the equation:

$$\begin{vmatrix} 110 - \omega^2 & -20 \\ -20 & 110 - \omega^2 \end{vmatrix} = (110 - \omega^2)^2 - 20^2 = 0$$

$$(110 - \omega^2 + 20)(110 - \omega^2 - 20) = 0$$

$$(130 - \omega^2)(90 - \omega^2) = 0$$

$$\omega_1 = \sqrt{130} = 11{,}402\ rad/s \quad \text{et} \quad \omega_2 = \sqrt{90} = 9{,}487 rad/s$$

Exercise 3:

a)- Equations of motion of the system

Let the displacements of block 1, block 2 and block 3 from their equilibrium positions be x_1, x_2 and x_3, respectively, each positive to the right. Our goal is to find the differential equations of motion of each block and then solve them to find $x_1(t), x_2(t)$ and $x_3(t)$. The Lagrangian of the system is: $\mathcal{L} = \mathrm{T} - \mathrm{V}$

with kinetic energy is: $T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2$

And potential energy is : $V = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2 + \frac{1}{2}kx_3^2$

The system is in translational motion along the x-axis and has three degrees of freedom x_1, x_2 and x_3 so Lagrange's equations give (Application of Lagrange's equations leads to):

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x_1}}\right) - \frac{\partial \mathcal{L}}{\partial x_1} = 0 & ① \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x_2}}\right) - \frac{\partial \mathcal{L}}{\partial x_2} = 0 & ② \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x_3}}\right) - \frac{\partial \mathcal{L}}{\partial x_3} = 0 & ③ \end{cases}$$

The equations of motion

$$\begin{cases} m\ddot{x}_1 + kx_1 - k(x_2 - x_1) = 0 \\ m\ddot{x}_2 + k(x_2 - x_1) - k(x_3 - x_2) = 0 \\ m\ddot{x}_3 + k(x_3 - x_2) + kx_3 = 0 \end{cases} \Rightarrow \begin{cases} m\ddot{x}_1 + 2kx_1 - kx_2 = 0 & ① \\ \ddot{m}x_2 + 2kx_2 - kx_1 - kx_3 = 0 & ② \\ m\ddot{x}_3 + 2kx_3 - kx_2 = 0 & ③ \end{cases}$$

b)- Natural vibration frequencies of the system.

In a normal mode, by definition each block oscillates with the same frequency, so we can simply try the solutions:

$$x_1(\mathrm{t}) = A\,cos(\omega t + \varphi) \quad / \quad \ddot{x}_1 = -\omega^2 x_1$$
$$x_2(\mathrm{t}) = B\,cos(\omega t + \varphi) \quad / \quad \ddot{x}_2 = -\omega^2 x_2$$
$$x_3(\mathrm{t}) = C\,cos(\omega t + \varphi) \quad / \quad \ddot{x}_3 = -\omega^2 x_3$$

with the same frequency for each.

By substituting these solutions and their second derivatives into equations ①, ② and ③, we find:

$$\begin{cases} -m\omega^2 A\,cos(\omega t + \varphi) + 2kA\,cos(\omega t + \varphi) - kB\,cos(\omega t + \varphi) = 0 \\ -m\omega^2 B\,cos(\omega t + \varphi) + 2kB\,cos(\omega t + \varphi) - kA\,cos(\omega t + \varphi) - kC\,cos(\omega t + \varphi) = 0 \\ -m\omega^2 C\,cos(\omega t + \varphi) + 2kC\,cos(\omega t + \varphi) - kB\,cos(\omega t + \varphi) = 0 \end{cases}$$

$$\begin{cases} (2k - m\omega^2)A - kB = 0 & ①' \\ -kA + (2k - m\omega^2)B - kC = 0 & ②' \\ -kB + (2k - m\omega^2)C = 0 & ③' \end{cases}$$

Which in matrix form becomes

$$\begin{pmatrix} (2k-m\omega^2) & -k & 0 \\ -k & (2k-m\omega^2) & -k \\ 0 & -k & (2k-m\omega^2) \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix}=0$$

Which have a nontrivial solution only if the determinant of the coefficient matrix is zero. That is, the secular equation is:

$$det_{sy}=\begin{vmatrix} (2k-m\omega^2) & -k & 0 \\ -k & (2k-m\omega^2) & -k \\ 0 & -k & (2k-m\omega^2) \end{vmatrix}$$

Expanding about the top row,

$$det_{sy}=(2k-m\omega^2)[(2k-m\omega^2)^2-k^2]+k[-k(2k-m\omega^2)]$$
$$=(2k-m\omega^2)[(2k-m\omega^2)-2k^2]$$

Factoring,

$$=(2k-m\omega^2)\left[(2k-m\omega^2)-\sqrt{2}k\right]\left[(2k-m\omega^2)+\sqrt{2}k\right]=0$$

The product of a linear and a quadratic equation in ω^2, with altogether three solutions. The first factor is zero if

$$\Rightarrow\ 2k-m\omega^2=0\Rightarrow\omega_1=\sqrt{\frac{2k}{m}}$$

The second factor is zero if $\left[(2k-m\omega^2)-\sqrt{2}k\right]\left[(2k-m\omega^2)+\sqrt{2}k\right]=0$which gives the other two eigenfrequencies

$$(2-\sqrt{2})k-m\omega^2=0\Rightarrow\omega_2=\sqrt{\frac{(2-\sqrt{2})k}{m}}$$

$$(2+\sqrt{2})k-m\omega^2=0\Rightarrow\omega_3=\sqrt{\frac{(2+\sqrt{2})k}{m}}$$

c)-The system has 3 modes (3 natural frequencies), so the solutions are:

$$x_{h1}(t)=A_1\cos(\omega_1 t+\varphi_1)+A_2\cos(\omega_2 t+\varphi_2)+A_3\cos(\omega_3 t+\varphi_3)$$
$$x_{h2}(t)=B_1\cos(\omega_1 t+\varphi_1)+B_2\cos(\omega_2 t+\varphi_2)+B_3\cos(\omega_3 t+\varphi_3)$$
$$x_{h3}(t)=C_1\cos(\omega_1 t+\varphi_1)+C_2\cos(\omega_2 t+\varphi_2)+C_3\cos(\omega_3 t+\varphi_3)$$

Now let's express the amplitudes B_i and C_i in terms of A_i, where $i=1,2\ and\ 3$

$$① '\Rightarrow B=\frac{(2k-m\omega^2)}{k}A\Rightarrow B_i=\frac{(2k-m\omega_i^2)}{k}A_i$$

$$B_1 = \frac{(2k - m\omega_1^2)}{k} A_1 = \frac{(2k - m\frac{2k}{m})}{k} A_1 = 0$$

$$B_2 = \frac{(2k - m\omega_2^2)}{k} A_2 = \frac{(2k - m\frac{(2-\sqrt{2})k}{m})}{k} A_2 = \sqrt{2} A_2$$

$$B_3 = \frac{(2k - m\omega_3^2)}{k} A_3 = \frac{(2k - m\frac{(2+\sqrt{2})k}{m})}{k} A_3 = -\sqrt{2} A_3$$

$$\textcircled{2}' \Rightarrow C = \frac{(2k - m\omega^2)B - kA}{k} \Rightarrow C_1 = -A_1, C_2 = A_2 \text{ et} C_3 = A_3$$

d) The equations of motion for the damped forced system (applying successively the fundamental dynamics relation to the three masses) are as follows:

$$\begin{cases} m\ddot{x}_1 + f\dot{x}_1 + 2kx_1 - kx_2 = F_0 \sin \omega_{ex} t & \textcircled{1}'' \\ m\ddot{x}_2 + -f\dot{x}_2 + 2kx_2 - kx_1 = 0 & \textcircled{2}'' \\ m\ddot{x}_3 + f\dot{x}_3 + 2kx_3 - kx_2 - kx_2 = 0 & \textcircled{3}'' \end{cases}$$

The solutions are given by:

$x_1(t) = x_{1h} + x_{1p}$, $x_2(t) = x_{2h} + x_{2p}$ et $x_3(t) = x_{3h} + x_{3p}$

$x_{1h,}\, x_{2h}$ et x_{3h}: $represent\ the\ homogeneous\ solutions\ (transient\ regime)$ and

$x_{1p,}\, x_{2p}$ et x_{3p}: $represent\ the\ particular\ solutions\ (steady-state\ regime)$.

To determine the steady-state solutions:

$x_{1p} = A\sin(\omega_{ex}t + \varphi_1)$, $x_{2p} = B\sin(\omega_{ex}t + \varphi_2)$ et $x_{3p} = C\sin(\omega_{ex}t + \varphi_3)$,

When using complex notation, we have : $x_{1p} = \bar{A}e^{j\omega_{ex}t}, x_{2p} = \bar{B}e^{j\omega_{ex}t}$ et $x_{3p} = \bar{C}e^{j\omega_{ex}t}$ et $F(t) = Im(F_0 e^{j\omega_{ex}t})$ avec $\bar{A} = Ae^{\varphi_1}$, $\bar{B} = Be^{\varphi_2}$ et $\bar{C} = Ce^{\varphi_3}$

By substituting these solutions and their derivatives into equations ①'' ②'' and ③'' on we obtain:

$$\begin{cases} [(2k - m\omega_{ex}^2) + jf\omega_{ex}]\bar{A} - k\bar{B} = F_0 \\ -k\bar{A} + [(2k - m\omega_{ex}^2) + jf\omega_{ex}]\bar{B} - k\bar{C} = 0 \\ -k\bar{B} + [(2k - m\omega_{ex}^2) + jf\omega_{ex}]\bar{C} = 0 \end{cases}$$

The friction coefficient $f = c = 1Nms^{-1}$, (le The main determinant $\mathfrak{D}_p = det_{sys}$).

$$\mathfrak{D}_p = det_{sys} = \begin{vmatrix} [(2k - m\omega_{ex}^2) + j\omega_{ex}] & -k & 0 \\ -k & [(2k - m\omega_{ex}^2) + j\omega_{ex}] & -k \\ 0 & -k & [(2k - m\omega_{ex}^2) + j\omega_{ex}] \end{vmatrix}$$

$$det_{sys} = [(2k - m\omega_{ex}^2) + j\omega_{ex}][(2k - m\omega_{ex}^2 + j\omega_{ex})^2 - 2k^2]$$

$$\bar{A} = \frac{\begin{vmatrix} F_0 & -k & 0 \\ 0 & [(2k - m\omega_{ex}^2) + j\omega_{ex}] & -k \\ 0 & -k & [(2k - m\omega_{ex}^2) + j\omega_{ex}] \end{vmatrix}}{det_{sys}}$$

$$\bar{A} = \frac{[(2k - m\omega_{ex}^2 + j\omega_{ex})^2 - k^2]F_0}{det_{sys}}$$

$$\bar{B} = \frac{\begin{vmatrix} [(2k - m\omega_{ex}^2) + j\omega_{ex}] & F_0 & 0 \\ -k & 0 & -k \\ 0 & 0 & [(2k - m\omega_{ex}^2) + j\omega_{ex}] \end{vmatrix}}{det_{sys}}$$

$$\bar{B} = \frac{F_0 k[(2k - m\omega_{ex}^2) + j\omega_{ex}]}{det_{sys}}$$

$$\bar{C} = \frac{\begin{vmatrix} [(2k - m\omega_{ex}^2) + j\omega_{ex}] & -k & F_0 \\ -k & [(2k - m\omega_{ex}^2) + j\omega_{ex}] & 0 \\ 0 & -k & 0 \end{vmatrix}}{det_{sys}}$$

$$\bar{C} = \frac{F_0 k^2}{det_{sys}}$$

$x_{1p} = A\sin(\omega_{ex}t + \varphi_1)$ avec $A = |\bar{A}|$ et $\varphi_1 = \frac{\mathbf{Im}\bar{A}}{|\bar{A}|}$ $\quad (\varphi_1 = arg\bar{A})$

$x_{2p} = B\sin(\omega_{ex}t + \varphi_2)$ avec $B = |\bar{B}|$ et $\varphi_2 = \frac{\mathbf{Im}\bar{B}}{|\bar{B}|}$ $\quad (\varphi_2 = arg\bar{B})$

$x_{3p} = C\sin(\omega_{ex}t + \varphi_3)$ avec $C = |\bar{C}|$ et $\varphi_3 = \frac{\mathbf{Im}\bar{C}}{|\bar{C}|}$ $\quad (\varphi_3 = arg\bar{C})$

Exercise 4:

The system is in translational motion along the x-axis and has two degrees of freedom x_1 and x_2. Additionally, m_1 mass is subjected to an excitation force, so the Lagrange equations can be written as:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x_1}}\right) - \frac{\partial \mathcal{L}}{\partial x_1} = F(t) & ① \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x_2}}\right) - \frac{\partial \mathcal{L}}{\partial x_2} = 0 & ② \end{cases}$$

The Lagrangian of the system is $\mathcal{L} = \mathrm{T} - \mathrm{V}$ with kinetic energy is :

$$T = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} m\dot{x}_2^2$$

And potentiel energy is :

$$V = \frac{1}{2} kx_1^2 + \frac{1}{2} k(x_2 - x_1)^2$$

On aboutit aux équations suivantes:

$$\begin{cases} m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) = F_0 \cos \omega_{ex} t \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) \quad = 0 \end{cases}$$

$$\Rightarrow \begin{cases} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = F_0 \cos \omega_{ex} t & ① \\ m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0 & ② \end{cases}$$

The solutions are:

$$x_1(t) = A_1 \cos(\Omega_1 t + \varphi_1) + A_2 \cos(\Omega_2 t + \varphi_2) + A' \cos(\omega_{ex} t + \theta_1)$$

$$x_2(t) = B_1 \cos(\Omega_1 t + \varphi_1) + B_2 \cos(\Omega_2 t + \varphi_2) + B' \cos(\omega_{ex} t + \theta_2)$$

Now let's determine the natural frequencies Ω_1 and Ω_2 . In a mode with a frequency Ω the solutions take the form:

$$x_1(\mathrm{t}) = A\, cos(\Omega t + \varphi) \text{et} \quad x_2(\mathrm{t}) = B\, cos(\Omega t + \varphi)$$

By replacing x_1 and x_2 and their second derivatives in the equations of motion ① and ②, we obtain the following equations:

$$\begin{cases} (-m_1\Omega^2 + k_1 + k_2)A - k_2B = 0 \\ \quad -k_2A + (-m_2\Omega^2 + k_2)B = 0 \end{cases}$$

$$det_{sys} = \begin{vmatrix} (-m_1\Omega^2 + k_1 + k_2) & -k_2 \\ -k_2 & (-m_2\Omega^2 + k_2) \end{vmatrix}$$

$$= (-m_1\Omega^2 + k_1 + k_2)(-m_2\Omega^2 + k_2) - k_2^2$$

$$det_{sys} = m_1m_2\Omega^4 - [m_2(k_1 + k_2) + m_1k_2]\Omega^2 + k_2(k_1 + k_2) - k_2^2 = 0$$

Ω_1 and Ω_2 are the positive roots of this equation.

$$B_1 = \frac{-m_1\Omega_1^2 + k_1 + k_2}{k_2} A_1 \quad et \quad B_2 = \frac{-m_1\Omega_2^2 + k_1 + k_2}{k_2} A_2$$

To simplify the calculations, let's take $m_1 = m_2 = m \quad et \quad k_1 = k_2 = k$

The determinant of the system becomes:

$det_{sys} = m^2\Omega^4 - 3km\Omega^2 + k^2 = 0$, equation of natural frequencies or "equation of eigenfrequencies".

We find: $\Omega_1 = \sqrt{\frac{3-\sqrt{5}}{2}}\sqrt{\frac{k}{m}}$ and $\Omega_2 = \sqrt{\frac{3+\sqrt{5}}{2}}\sqrt{\frac{k}{m}}$

$$B_1 = \frac{\sqrt{5}+1}{2} A_1 \quad et \quad B_2 = \frac{\sqrt{5}-1}{2} A_2$$

$$x_{1h}(t) = A_1 \cos((\sqrt{\frac{3-\sqrt{5}}{2}}\sqrt{\frac{k}{m}})t + \varphi_1) + A_2 \cos((\sqrt{\frac{3+\sqrt{5}}{2}}\sqrt{\frac{k}{m}})t + \varphi_2)$$

$$x_{2h}(t) = \frac{\sqrt{5}+1}{2} A_1 \cos((\sqrt{\frac{3-\sqrt{5}}{2}}\sqrt{\frac{k}{m}})t + \varphi_1) + \frac{\sqrt{5}-1}{2} A_2 \cos((\sqrt{\frac{3+\sqrt{5}}{2}}\sqrt{\frac{k}{m}})t + \varphi_2)$$

To determine the steady-state solutions :

$$x_{1p}(\mathrm{t}) = A' \cos(\omega t + \theta_1) \text{ et } x_{2p}(\mathrm{t}) = B' \cos(\omega t + \theta_2)\ ,$$

We are looking for the constants $A', B,' \theta_1$ et θ_2

For this, we use the complex number method. Therefore, let's set:

$$x_{1p} = A' e^{j(\omega_{ex}t + \theta_1)} = \overline{A'} e^{j\omega_{ex}t} \quad \text{avec} \quad \overline{A'} = A' e^{j\theta_1}$$

$$x_{2p} = B' e^{j(\omega_{ex}t + \theta_2)} = \overline{B'} e^{j\omega_{ex}t} \quad \text{avec} \quad \overline{B'} = B' e^{j\theta_2}$$

and $$F(t) = \mathcal{R}(F_0 e^{j\omega_{ex}t})$$

By substituting these notations into the system of equations ① and ②, we find:

$$\begin{cases} (2k - m\omega_{ex}^2)\overline{A'} - k\overline{B'} = F_0 \\ -k\overline{A'} + (k - m\omega_{ex}^2)\overline{B'} = 0 \end{cases}$$

$$\overline{A'} = \frac{(k - m\omega_{ex}^2)F_0}{(2k - m\omega_{ex}^2)(k - m\omega_{ex}^2) - k^2} \text{ et } \overline{B'} = \frac{kF_0}{(2k - m\omega_{ex}^2)(k - m\omega_{ex}^2) - k^2}$$

With

$$A' = |\overline{A'}|,\ B' = |\overline{B'}|\ ,\quad \theta_1 = Arg\overline{A'} \text{ et } \theta_2 = Arg\overline{B'}.$$

$$\theta_1 = \theta_2 = 0$$

The steady-state solutions can be expressed as : $x_{1p} = \overline{A'} \cos\omega_{ex}t$ et $x_{2p} = \overline{B'} \cos\omega_{ex}t$

b)-The system is now subjected to a frictional force assumed to be proportional to the velocity, applying the fundamental principle of dynamics:

$$m\vec{\gamma} = \Sigma\overrightarrow{F_{ext}} = \vec{R} + \vec{P} + \overrightarrow{F_r} + \overrightarrow{F_f} + \overrightarrow{F_{ex}} \; .$$

After projection onto the x-axis, the equations of motion are written as:

$$\begin{cases} m_1\ddot{x}_1 + f\dot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = F_0 \cos\omega_{ex}t \\ m_2\ddot{x}_2 + f\dot{x}_2 - k_2x_1 + k_2x_2 = 0 \end{cases}$$

The steady-state solutions:

$x_{1p} = A\cos(\omega_{ex}t + \alpha_1)$ et $x_{2p} = B\cos(\omega_{ex}t + \alpha_2)$ are expressed in complex notation as:

$x_{1p} = \bar{A}\, e^{j(\omega_{ex}t)}$ et $x_{2p} = \bar{B}e^{j(\omega_{ex}t)}$, $\bar{A} = Ae^{j\alpha_1}, \bar{B} = Be^{j\alpha_2}$

We obtain:

$$\begin{cases} (-m_1\omega_{ex}^2 + jf\omega_{ex} + k_1 + k_2)\bar{A} - k_2\bar{B} = F_0 \\ -k_2\bar{A} + (-m_2\omega_{ex}^2 + k_2 + jf\omega_{ex})B = 0 \end{cases}$$

So

$$\bar{A} = \frac{(-m_2\omega_{ex}^2 + k_2 + jf\omega_{ex})F_0}{(-m_1\omega_{ex}^2 + jf\omega_{ex} + k_1 + k_2)(-m_2\omega_{ex}^2 + k_2 + jf\omega_{ex}) - {k_2}^2}$$

and

$$\bar{B} = \frac{k_2F_0}{(-m_1\omega_{ex}^2 + jf\omega_{ex} + k_1 + k_2)(-m_2\omega_{ex}^2 + k_2 + jf\omega_{ex}) - {k_2}^2}$$

For $f = c = 1N.ms^{-1}$

$$\bar{A} = \frac{(-m_2\omega_{ex}^2 + k_2 + j\omega_{ex})F_0}{(-m_1\omega_{ex}^2 + j\omega_{ex} + k_1 + k_2)(-m_2\omega_{ex}^2 + k_2 + j\omega_{ex}) - {k_2}^2}$$

$$\bar{B} = \frac{k_2F_0}{(-m_1\omega_{ex}^2 + j\omega_{ex} + k_1 + k_2)(-m_2\omega_{ex}^2 + k_2 + j\omega_{ex}) - {k_2}^2}$$

$$A = |\bar{A}|, \qquad B = |\bar{B}|$$

$$\alpha_1 = \alpha_{1n} - \alpha_{1d}$$

$$\alpha_2 = \alpha_{2n} - \alpha_{2d}$$

Exercise 5:

a) Let's consider the semi-system constituted by the order pendulums (n-1), n et (n+1)

$$L = T - V$$

kinetic energy : $T = \frac{1}{2}I\dot{\theta}_{n-1}^2 + \frac{1}{2}I\dot{\theta}_n^2 + \frac{1}{2}I\dot{\theta}_{n+1}^2$

With $I = ml^2$

Potential energy : $V = -mgh_{n-1} - mgh_n - mgh_{n+1} + V'$

V' represents the potential energy of deformation of the springs.

For small angles θ_n :

$$V' = \frac{1}{2}k(l\theta_n - l\theta_{n-1})^2 + \frac{1}{2}k(l\theta_n - l\theta_{n+1})^2$$

Therefore :

$$L = \frac{1}{2}I\dot{\theta}_{n-1}^2 + \frac{1}{2}I\dot{\theta}_n^2 + \frac{1}{2}I\dot{\theta}_{n+1}^2 - \frac{1}{2}k(l\theta_n - l\theta_{n-1})^2 - \frac{1}{2}k(l\theta_n - l\theta_{n+1})^2 + mglcos\theta_{n-1} + mglcos\theta_n + mgl\theta_{n+1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_n}\right) - \left(\frac{\partial L}{\partial \theta_n}\right) = ml^2\ddot{\theta}_n + mglsin\theta_n + kl^2(\theta_n - \theta_{n-1}) + kl^2(\theta_n - \theta_{n+1}) = 0$$

$$\ddot{\theta}_n + \frac{g}{l}sin\theta_n + \frac{2k}{m}\theta_n - \frac{k}{m}(\theta_{n+1} + \theta_{n-1}) = 0$$

For small angles θ_n (n=0,………………..N) , $sin\theta_n \approx \theta_n$ et $x_n \approx l\theta_n$

Hence, the equation of motion in x is:

$$\ddot{x}_n + \left(\frac{g}{l} + \frac{2k}{m}\right)x_n - \frac{k}{m}(x_{n+1} + x_{n-1}) = 0$$

b) $x_n = (AsinKna + BcosKna)\cos(\omega t + \varphi)$

$$x_{n+1} = (AsinK(n+1)a + BcosK(n+1)a)\cos(\omega t + \varphi)$$

$$x_{n-1} = (AsinK(n-1)a + BcosK(n-1)a)\cos(\omega t + \varphi)$$

Knowing that:

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha.\cos\beta$$

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha.\sin\beta$$

$$x_{n+1} + x_{n-1} = 2cosKa(sinKna + BcosKna)\cos(\omega t + \varphi) = 2x_n cosKa$$

$$\ddot{x}_n = -\omega^2 x_n$$

By substituting these expressions into the equation of motion, we obtain

$$-\omega^2 + \frac{g}{l} + \frac{2k}{m} - \frac{2k}{m}cosKa = 0 \Longrightarrow \omega^2 = \frac{g}{l} + \frac{2k}{m}(1 - cosKa)$$

$$(cosKa = 1 - 2sin^2\frac{Ka}{2})$$

$$\omega = \sqrt{\frac{g}{l} + \frac{4k}{m}sin^2\frac{Ka}{2}}$$

c) The boundary conditions are :

$x_0 = 0$ et $x_{N-1} = 0$

$x_0 = 0 + B = 0 \Longrightarrow B = 0$ donc $x_n = AsinKna \cos(\omega t + \varphi)$

d) $x_{N-1} = AsinK(N-1)a \cos(\omega t + \varphi)$

$$x_{N-1} = 0 \Longrightarrow sinK(N-1)a = 0 \Longrightarrow K(N-1) = m\pi$$

$$sina = sinb \Longrightarrow a = b + 2K\pi, \quad a = \pi - b + 2K\pi$$

$$m = 0, 1, 2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots., (N-1)$$

The possible values of K are:

$$K = \frac{m\pi}{(N-1)a}$$

With

$$m = 1, 2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots., (N-2)$$

$$\omega_{min} = \sqrt{\frac{g}{l} + \frac{4k}{m}sin^2\frac{K_1 a}{2}} = \sqrt{\frac{g}{l} + \frac{4k}{m}sin^2\frac{\pi}{2(N-1)}}$$

$$\omega_{max} = \sqrt{\frac{g}{l} + \frac{4k}{m}sin^2\frac{K_{N-2} a}{2}} = \sqrt{\frac{g}{l} + \frac{4k}{m}sin^2\frac{(N-2)\pi}{2(N-1)}}$$