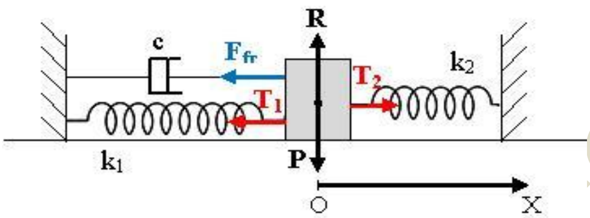
University of Badji Mokhtar-Annaba Faculty of Engineering Sciences Department of Science and Technology, 2nd Year Pr. L. BECHIRI Module: Physics 3		2025-2026 Tutorial 2 Solutions

Exercise 1 :

Consider the following mechanical system

$$m = 1\ \text{kg}\ , k_1 = 20\ \text{N/m}\ ;\ k_2 = 5\ \text{N/m}$$



The existence of a piston → a viscous friction force $\mathbf{F_{fr}}$ of the form:

$\overrightarrow{F_f} = --c\vec{v} = -c\dot{x}\vec{\imath}\quad (v = \frac{dx}{dt} = \dot{x})$

a) *Equation of motion for mass m along the X-axis*

(Newton's law of motion is used to establish the differential equation of motion of one-degree-of-freedom systems).

$$m\vec{\gamma} = \Sigma\overrightarrow{F_{ext}} = \overrightarrow{F_{r1}} + \overrightarrow{F_{r2}} + \overrightarrow{F_f} \qquad (\gamma = \frac{d^2x}{dt^2} = \ddot{x} = acceleration)$$

$\overrightarrow{F_{r1}} = \vec{T}_1 \qquad , \overrightarrow{F_{r2}} = \vec{T}_2$

Along the (OX) axis: $\vec{T}_1 + \vec{T}_2 + \overrightarrow{F_f} = m\ddot{x}\vec{\imath}$

(The projection of $\vec{P}$ and $\vec{R}$ along (OX) is zero ; $\ddot{x} = \frac{d^2x}{dt^2}$; $\vec{\imath}$: unit vector along the (OX) axis)

$\Rightarrow \quad m\ddot{x} = -k_1x - k_2x - c\dot{x} \Rightarrow m\ddot{x} + c\dot{x} + (k_1 + k_2)x = 0$

Dividing by m, we obtain a second-order differential equation of the form:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{(k_1 + k_2)}{m}x = 0$$

$\ddot{x} + \frac{c}{m}\dot{x} + \frac{(k_1+k_2)}{m}x = 0$ avec $\omega_0^2 = \frac{(k_1+k_2)}{m} = 25\ rad/s$

So the differential equation of motion is: $\ddot{x} + c\dot{x} + 25x = 0$

b)The values of c for which we have damped oscillatory motion:

The characteristic equation of this second-order differential equation is:

$r^2 + \mathrm{c}r + 25 = 0 \Leftrightarrow ar^2 + b\,\mathrm{r} + c = 0$; with $\Delta = \mathrm{b}^2 - 4\mathrm{ac} = \mathrm{c}^2 - 4\mathrm{x}25$

For the motion to be damped oscillatory, it is necessary for Δ to be less than zero ($\Delta < 0$).

Therefore, the values of c are: $\mathrm{c}^2 - 100 < 0 \quad \Rightarrow \quad c < 10\ \mathrm{Nsm}^{-1}$.

c) The general expression of x(t) is:

We have : $c = 0.1\ \mathrm{Nsm}^{-1} \Rightarrow \Delta < 0$ so we have damped oscillatory motion..

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{(k_1 + k_2)}{m}x = 0$$

Let's define $2\,\lambda = \frac{c}{m}$ and $\omega_0^2 = \frac{(k_1+k_2)}{m}$

We obtain the differential equation:: $\ddot{x} + 2\,\lambda\dot{x} + \omega_0^2 x = 0$

If $\Delta' = b'^2 - ac = \lambda^2 - \omega_0^2 = \mathrm{j}^2(\omega_0^2 - \lambda^2) = \mathrm{j}^2\omega_a^2$

, then we have two complex roots: $r_{1,2} = -\lambda \pm \mathrm{j}\omega_a$ where $\omega_a = \sqrt{\omega_0^2 - \lambda^2}$

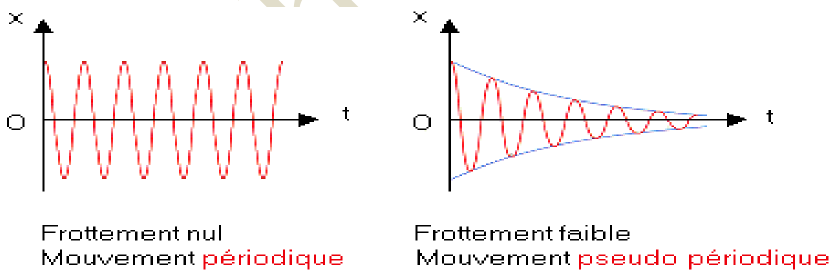
Therefore, the solution to the differential equation is of the form $x(t) = Ae^{-\lambda t}\cos(\omega_a t + \varphi)$

Here : $2\,\lambda = \frac{c}{m}$ et $\omega_a = \sqrt{\frac{(k_1+k_2)}{m} - (\frac{c}{2m})^2}$

ω_0 : natural frequency; ω_a pseudo-frequency; λ : damping factor

.Given values : $\lambda = 0.05$ $\omega_0 = 5$ rad/s; $\omega_a = \sqrt{25 - (0.05)^2} = 4.99 \cong 5 rad/s$

So, the solution becomes: $x(t) = Ae^{-0.05t}\cos(5t + \varphi)$



- Logarithmic decrement δ

$$\delta = ln\left[\frac{x(t)}{x(t+T)}\right] = \lambda \mathrm{T_a}$$

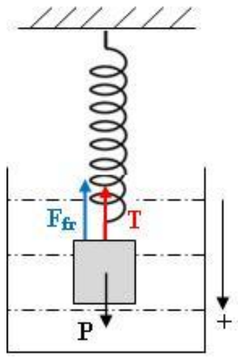
Where $\mathrm{T_a}$ *is the pseudo-period, and in this case* $\delta = \lambda\left(\frac{2\pi}{\omega_a}\right) = 2\pi\frac{0.05}{5} = 2\pi\mathrm{x}10^{-2}$

- Quality factor Q $Q = \frac{\omega_0}{2\lambda}$

For the given values: $Q = \frac{5}{2\mathrm{x}0.05} = 50$

Exercise 2 :

Consider the following mechanical system:



$$m = 1kg,\ \mathrm{c}\ =\ 0.4\ \mathrm{Nms}^{-1},\ \delta\ =\ 10^{-2};\ K = ?$$

We have a viscous friction force: $\vec{F}_f = -c\vec{v}$

($\vec{v}$: velocity; ; $v_x = \dot{x} = \frac{dx}{dt}$ velocity in the $\boldsymbol{x}-$ direction)

Note: The mass is in translational motion along the OX axis. The problem can be solved using two methods:

1-Using : $m\vec{\gamma} = \sum \vec{F}_{ex}$ ($\gamma_x = \ddot{x} = \frac{d^2x}{dt^2}$: acceleration)

$$m\ddot{x} = P + T + F_f$$

$$m\ddot{x} = -c\dot{x} - kx$$

Given that : $mg - k\Delta l = 0$ (equilibrium condition))

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

2-Writing the Lagrangian of the system (to be detailed below).

So, the Lagrange equation is::

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = -\frac{\partial D}{\partial \dot{x}} \qquad (1)$$

D is the dissipation function.

The Lagrangian L is given by : $L = T - V$ ($T = \frac{1}{2}m\dot{x}^2$: the kinetic energy ; $V = \frac{1}{2}kx^2$: the potential energy) and $D = \frac{1}{2}c\dot{x}^2$: dissipated energy (The loss of energy over time by a dynamic system, primarily attributed to friction.).

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \qquad \frac{\partial L}{\partial x} = -kx \qquad \frac{\partial D}{\partial \dot{x}} = c\dot{x}$$

Substituting into equation (1), we get:

$$m\ddot{x} + c\dot{x} + kx = 0$$

Dividing by m, we obtain a second-order differential equation of the form:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \quad \Leftrightarrow \quad \ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0$$

a) ***The values of k for which we have a damped oscillatory motion:***

This linear, second-order, homogeneous, differential equation has a solution of the form:

$$x(t) = Ae^{-rt}\ (trial\ solution)$$

$$r^2 + \frac{c}{m}r + \frac{k}{m} = 0$$

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0$$

$$ar^2 + br + c = 0$$

With $\Delta = b^2 - 4ac = 4\lambda^2 - 4\omega_0^2$

For damped oscillatory motion, ($\Delta < 0$)..

Therefore, the values of k are given by:

$$\lambda^2 - \omega_0^2 < 0 \quad \Rightarrow \quad \lambda^2 < \omega_0^2 \quad \Rightarrow \quad \left(\frac{c}{2m}\right)^2 < \frac{k}{m}$$

Hence :

$k > \frac{c^2}{4m}$ A.N $k > 0.04N/m$

b) Determination of the pseudo-period T_a and the stiffness constant k :

$The\ logarithmic\ decrement\ is\ given\ by\ \delta = \lambda T_a$

On a : $\delta = \lambda T_a$ $T_a = \frac{\delta}{\lambda}$ donc $T_a = \frac{2m\delta}{c} \Longrightarrow$ $A.N: T_a = 0.05s$

The pseudo-frequency ω_a is given by: :

$$\omega_a^2 = \omega_0^2 - \lambda^2 \Longrightarrow \omega_0^2 = \frac{k}{m} = \omega_a^2 + \lambda^2 \Longrightarrow k = m(\omega_a^2 + \lambda^2)$$

$$k = m(\omega_a^2 + \lambda^2)$$

$$\lambda^2 = \frac{c^2}{4m^2} \quad \text{et} \quad \omega_a^2 = (\frac{2\pi}{T_a})^2$$

A. N.: $k = 1.57\ 10^4$ N/m

Exercise 3 :

$c = 50Ns/m, \quad k = 600N/m,\ m = 7\ kg\ , \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{600}{7}} = 9.258\ rad/s$

$$c_c = 2m\omega_n = 2(7)(9.258) = 129.6N.s/m$$

Since $\zeta = \frac{c}{c_c} = \frac{50}{129.6} = 0.39 < 1$ $c < c_c$, the system is underdamped,

$\omega_a = \omega_n\sqrt{1-\left(\frac{c}{c_c}\right)^2} = 9.258\sqrt{1-\left(\frac{50}{129.6}\right)^2} = 8.542\ rad/s$

$$\lambda = \frac{c}{2m} = \frac{50}{2(7)} = 3.751$$

From the equation

$$x(t) = Ae^{-\lambda t}\sin\left(\omega_a t + \varphi\right)$$

Where A and φ are constants generally determined from the initial conditions of the problem

$$x(t) = Ae^{-\frac{c}{2m}t}\sin\left(\omega_a t + \varphi\right)$$

Or $v = \dot{x}(t) = A\left[-e^{-\frac{c}{2m}t}\omega_a\cos\left(\omega_a t + \varphi\right) + (-\frac{c}{2m})\ e^{-\frac{c}{2m}t}\sin\left(\omega_a t + \varphi\right)\right]$

$$v = A\left[-e^{-\frac{c}{2m}t}\omega_a\cos\left(\omega_a t + \varphi\right) + (-\frac{c}{2m})\ e^{-\frac{c}{2m}t}\sin\left(\omega_a t + \varphi\right)\right]$$

$$v = Ae^{-\frac{c}{2m}t}\left[\omega_a\cos\left(\omega_a t + \varphi\right) - \frac{c}{2m}\sin\left(\omega_a t + \varphi\right)\right]$$

Let's apply the initial conditions to evaluate the integration constants: at $t = 0, x = 0\ et\ \ v = -0.6m/s$

$0 = Ae^{\,0}\sin\left(0 + \varphi\right)$ comme $A \neq 0$

$$\sin\varphi = 0, \quad \varphi\ = 0$$

$$-0.6 = Ae^{-0}[\omega_a\cos\left(0 + \varphi\right) - 0]$$

$$-0.6 = Ae^{-0}[8.542\cos\left(0\right) - 0]$$

$$A = -0.07002m$$

$$x(t) = -0.07002e^{-3.751t}\sin 8.542t\ \ m$$

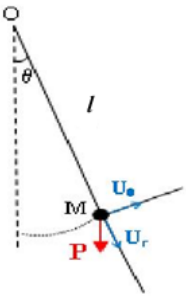
Exercise 4 :

A uniform rod (of length l and negligible mass) carrying at its end a mass m considered as point-like.

a) *Equation of motion for m*

The articulation at O is perfect, → implying a free system (undamped). *Fex* is considered.

The angular momentum theorem $\sigma_{/o}$ is employed:



$$\frac{d\vec{\sigma}_0}{dt} = I\ddot{\theta}\vec{k} = \sum \vec{\mathcal{M}}(\overrightarrow{F_{ex}}/0) \qquad (1)$$

$\vec{k}$ **: perpendicular to the vertical plane** ; $I = ml^2$: moment of inertia

$\vec{\mathcal{M}}(\vec{F}/0)$: moment of external forces; $\ddot{\theta} = \frac{d^2\theta}{dt^2}$: angular acceleration.

We calculate the moment of $\vec{P}$:

$$\vec{\mathcal{M}}\ (\vec{P}/0) = \overrightarrow{OM}\wedge\vec{P} = |OM||P|\sin\,(\overrightarrow{OM},\vec{P})\vec{k}$$

$$\overrightarrow{OM} = l\vec{U}_r$$

$$\vec{P} = \text{mg}\cos\theta.\vec{U}_r - \text{mg}\sin\theta\,\vec{U}_\theta$$

So: $\vec{\mathcal{M}}\ (\vec{P}/0) = -mglsin\theta\vec{k}$

Substituting this into equation (1), we obtain:

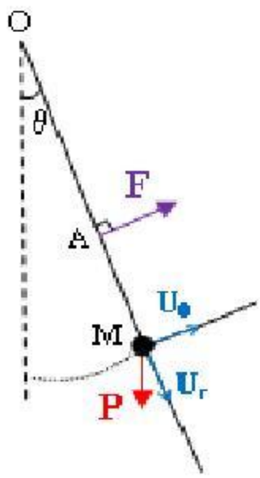
$I\ddot{\theta} = -\text{mgl}\sin\theta$, For small amplitude oscillations ($sin\theta \approx \theta$)

We divide by $I = ml^2$ (moment of inertia for a point mass, where l represents the distance between the mass and the axis of rotation (see Huygens' theorem for other shapes)

Thus, we get : $\ddot{\theta} + \frac{g}{l}\theta = 0$

This is a second-order differential equation of the form:

$$\ddot{\theta} + \omega_0^2\theta = 0 \quad \text{where} \quad \omega_0 = \sqrt{\frac{g}{l}} \quad ,$$



The general expression for θ(t) is in the form:

$$\theta(t) = Acos\left(\sqrt{\frac{g}{l}}\,t + \varphi\right)$$

b) *The rod is subjected to an external force $\vec{F}$*

We have : $\vec{\mathcal{M}}\ (\vec{P}\ /0) + \vec{\mathcal{M}}\ (\vec{F}\ /0) = I\ddot{\theta}\vec{k}$ (2)

$$I\ddot{\theta}\vec{k} = (\overrightarrow{OM}\wedge\vec{P} + \overrightarrow{OA}\wedge\vec{F}).\vec{k}$$

$\vec{F} = F\vec{U}_\theta$ et $\overrightarrow{OA} = a\vec{U}_r$

Let's calculate the moment of $\vec{F}$:

$$\vec{\mathcal{M}}\ (\vec{F}/0) = \ \overrightarrow{OA}\wedge\vec{F} = |OA||F|\sin\,(\overrightarrow{OA},\vec{F})\vec{k}$$

Since $\vec{F}$ is perpendicular to the rod at point A (i.e . $\vec{F} \perp \overrightarrow{OA} \Rightarrow \sin(\overrightarrow{OA}, \vec{F}) = 1$)

So $\vec{\mathcal{M}}\ (\vec{F}/0) = \ aF\vec{k}$

Substituting into (2) and dividing by I, the equation of motion becomes:

$$\ddot{\theta} + \frac{g}{l}\theta = \frac{aF}{ml^2}$$

This is a second-order differential equation with a forcing term (forced system).

c) *La The mass is subjected to a viscous friction force $\vec{F_f}$:*

$$I\ddot{\theta}\vec{k} = (\overrightarrow{OM}\wedge\vec{P} + \overrightarrow{OM}\wedge\vec{F_f}).\vec{k} \qquad (3)$$

Where $\vec{F_f} = -c\vec{v}$ où $\vec{v}$: represents a linear velocity tangential to the trajectoryet $\vec{F_f}$ is perpendicular to the rod at point M (i.e $\vec{F_f} \perp \overrightarrow{OM}$ **which implies** $\sin\,(\overrightarrow{OM}, \vec{F_f}) = 1$)

Let's calculate the moment of $\vec{F_f}$:

$$\vec{\mathcal{M}}\ (\vec{F_f}\ /0) = \overrightarrow{OM} \wedge \vec{F_f}\ = |OM||F_f|\sin\,(\overrightarrow{OM},\vec{F_f})\vec{k}$$

$$\vec{\mathcal{M}}\ (\vec{F_f}\ /0) = \overrightarrow{OM} \wedge - c\vec{v}_m$$

So: $\vec{\mathcal{M}}\ (\vec{F_f}\ /0) = -cl^2\dot{\theta}\vec{k}$

We have $x = l\theta$ (knowing that for small amplitude oscillations, l represents the radius of the arc). The velocity $v = l\dot{\theta}$

Substituting into equation (3) and dividing by J, the equation of motion becomes:

$$\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{g}{l}\theta = 0$$

This is a second-order differential equation of the form:

$$\ddot{\theta} + 2\lambda\dot{\theta} + \omega_0^2\theta = 0$$

$$\omega_a^2 = \omega_0^2 - \lambda^2$$

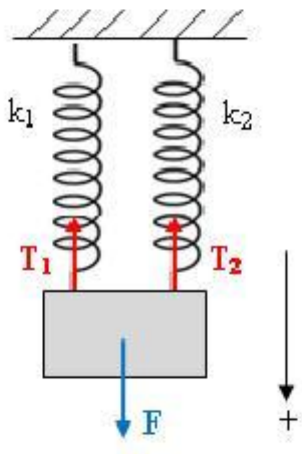
With $\omega_a = \sqrt{\omega_0^2 - \lambda^2} = \sqrt{\frac{g}{l} - \frac{c^2}{4m^2}}$ ((indicating a damped free system).).

The general expression for θ(t) is in the form:

$$\theta(t) = Ae^{-\lambda t}cos(\,\omega_a\,t + \varphi\,)$$

Exercise 5 :

a) **The equation of motion for the mass**



At equilibrium : $\sum \vec{F} = \vec{0}$

$$\vec{P} + \vec{F_1} + \vec{F_2} = \vec{0}$$

$$mg - k_1\,\Delta l_0 - k_2\,\Delta l_0 = 0$$

$$mg - (k_1 + k_2)\Delta l_0 = 0 \qquad (1)$$

In motion: $\sum \vec{F}_{ex} = m\vec{\gamma}$

$$\vec{P} + \vec{F}_{r1} + \overrightarrow{F_{r2}} + \overrightarrow{F(t)} = m\overrightarrow{\gamma}$$

$\vec{F}_{r1} = \vec{T_1}$, $\vec{F}_{r2} = \vec{T_2}$

$$m\ddot{x} = mg - k_1(\Delta l_0 + x) - k_2(\Delta l_0 + x) + F(t)$$

$$m\ddot{x} = -(k_1 + k_2)x - ((k_1 + k_2)\Delta l_0 + mg + F(t)$$

After using the equilibrium condition (equation 1), the equation becomes:

$$m\ddot{x} = -(k_1 + k_2)x + F(t)$$

$$m\ddot{x} + (k_1 + k_2)x = +F(t)$$

Dividing by m:

$$\ddot{x} + \frac{(k_1 + k_2)}{m}x = \frac{0.2}{m}sin3t$$

Given that $m = 10\text{ kg},\ k_1 = 30\frac{\text{N}}{\text{m}},\ k_2 = 70\frac{\text{N}}{\text{m}}$

the equation becomes: $\boldsymbol{\ddot{x} + 10x = 0.02sin3t}$**......................(*)**

b) The general solution of this differential equation (*) is in the form:

$$x_g(t) = x_h(t) + x_p(t)$$

Where $x_h(t)$ (t) is the homogeneous solution of the equation without the forcing term (*):

$$\ddot{x} + 10x = 0$$

This has the form: $\ddot{x} + \omega_0^2 x = 0$

By identification, we obtain:

$$\omega_0^2 = 10 \Rightarrow \omega_0 = \sqrt{10}\text{ rad/s}$$

$$x_h(t) = A\cos(\omega_0\,t + \varphi)$$

$$x_h(t) = A\cos(\sqrt{10}\,t + \varphi)$$

And $x_p(t)$ is the particular solution with the same form as the excitation force:

$x_p(t) = \text{B}\sin(3t + \theta)$

To determine $x_p(t)$ complex numbers are used:

$$\underline{x_p}\ (t)\ =\ \left(Be^{j(3t+\theta)}\right) = \overline{B}e^{j3t}$$

The expression $\overline{B}\ =\ Be^{j\theta}$ represents the complex amplitude of the oscillator. Here:

- B is the magnitude (module) of the complex amplitude,
- θ is the argument of the complex amplitude.

For the excitation force, we obtain: $Im(0.02e^{j3t})$

$\underline{x_p}\ (t)$ is the particular solution of the equation (*) : $\ddot{x} + \mathbf{10}x = \mathbf{0.02}sin\mathbf{3}t$

and its derivatives are given as:

$\underline{\dot{x}_p}(t) =\ 3j\overline{B}e^{j3t},\ \ \underline{\ddot{x}_p}(t) = -9\overline{B}e^{j3t}\ , j^2\ = -1$

Now, we substitute $\underline{x_p}\ (t),\ \underline{\dot{x}_p}(t),$ and $\underline{\ddot{x}_p}(t)$ into the equation (*) :

$$-9\overline{B}\,e^{j3t}\ + 10\overline{B}\,e^{j3t}\ = 0.02e^{j3t}$$

$$\overline{B}\ = 0.02$$

$\overline{B}$ is real, so $|\overline{B}|$ = B = 0.02

($\overline{B}$ has no imaginary part. In this case, the complex amplitude is purely real, and its magnitude $|\overline{B}|$ is equal to its real part B, which is 0.02).

$$tg\theta = \frac{ImB}{ReelB} = \frac{0}{0.02} = 0$$

$tg\theta = 0 \Rightarrow \theta = 0$

$$x_p(t) = 0.02\ sin3t$$

$$x_g(t) =\ \ x_h(t) + x_p(t)$$

$$x_g(t) = A\cos\left(\sqrt{10}\,t + \varphi\right) + 0.02\ \text{sin3t}$$

at t = 0, the initial conditions are: $\begin{cases} x = 5cm \\ \dot{x}\ = 0 \end{cases}$ yielding:

$$x_g(t) = A\cos\left(\sqrt{10}\,t + \varphi\right) + 0.02\ \text{sin3t}$$

$$\dot{x}_g(t) =\ -A\sqrt{10}\sin\left(\sqrt{10}\,t + \varphi\right) + 0.02\text{x3.cos3t}$$

$$\begin{cases} x(0) = Acos\varphi = 0.05 & (1) \\ \dot{x}(0) = -A\sqrt{10}\sin(\varphi) + 0.06 = 0 & (2) \end{cases}$$

$$\frac{(2)}{(1)} \Rightarrow \text{tg}\varphi = \frac{3\sqrt{10}}{25} = 0.379 \Rightarrow \varphi = 20.78^0$$

According to equation (1), the amplitude A is determined by:

$$A = \frac{0.05}{cos\varphi} = 5.35cm$$

Therefore, the general solution for the motion of the system $x_g(t)$ is given by:

$$x_g(t) = 5.35\cos(\sqrt{10}\, t + 0.36) + 2\,\sin 3t \qquad [cm]$$

<u>Exercise 6</u> :

a) <u>The natural frequency of the system's oscillations:</u>

$$\sum \vec{F} = m\vec{\gamma} \quad \Longrightarrow \quad m\vec{\gamma} = \vec{P} + \vec{R} + \vec{F_r}$$

The projection of the movement in x is: $m\ddot{x} = -kx$

$$m\ddot{x} + kx = 0$$

Dividing by m: $\ddot{x} + \frac{k}{m}x = 0$

It is in the form: $\ddot{x} + \omega_0^2 x = 0$

By identification, we obtain: $\omega_0^2 = \frac{k}{m} \Rightarrow \omega_0 = \sqrt{\frac{k}{m}}$

Numerical applications : $\omega_0 = \sqrt{\frac{9{,}10^2}{1}} = 30\text{rad/s}$

b) $x(t) = A\cos(\omega_0\, t + \varphi)$

à t = 0 $\begin{cases} x = 4cm \\ \dot{x} = 0 \end{cases}$

$$x(t) = A\cos(\omega_0\, t + \varphi)$$

$\dot{x}(t) = -A\omega_0 \sin(\omega_0\, t + \varphi)$

$$\begin{cases} x(0) = Acos\varphi = 4cm & (1) \\ \dot{x}(0) = -A\omega_0 \sin(\varphi) = 0 & (2) \end{cases}$$

$$(2) \Rightarrow \sin\varphi = 0 \Rightarrow \varphi = 0$$

According to (1): $A = \frac{4}{cos0} = 4$ cm

$$x(t) = 4cos30t \quad [cm]$$

c) $\overrightarrow{F_f} = -\,c\,\vec{v}$

$$\sum \vec{F} = m\vec{\gamma}$$

$$m\vec{\gamma} = \vec{P} + \vec{R} + \vec{F_r} + \overrightarrow{F_f}$$

$$m\ddot{x} = -c\dot{x} - kx$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

Dividing by m: $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$

It is in the form: $\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0$

By identification, we obtain:

$$2\lambda = \frac{c}{m} \Rightarrow \lambda = \frac{c}{2m}, \quad \omega_0^2 = +\frac{k}{m} \Rightarrow \omega_0 = \sqrt{\frac{k}{m}}$$

The damped natural frequency:

$$\omega_a^2 = \omega_0^2 - \lambda^2$$

$$\omega_a^2 = \omega_0^2 - \left(\frac{c}{2m}\right)^2$$

$$\left(\frac{c}{2m}\right)^2 = \omega_0^2 - \omega_a^2 \Longrightarrow c = 2m\sqrt{\omega_0^2 - \omega_a^2}$$

$$c = 2\sqrt{\omega_0^2 - \omega_a^2} = 2\sqrt{9x10^2 - 5.45^2} \cong 59 \text{ N.s.}m^{-1}$$

d)The logarithmic decrement:

$$\delta = \lambda T_a$$

$$\delta = \frac{c}{2m}\frac{2\pi}{\omega_a} = 34$$

$e)$ *System quality factor*

$$Q = \frac{2\pi}{1 - e^{-2\delta}} = \frac{2\pi}{1 - 1 + 2\lambda T_a} = \frac{\pi}{\lambda T_a} = \frac{\omega_0}{2\lambda} = \frac{30}{\frac{c}{m}} = \frac{30}{59} = 0.50$$

For weak friction.

$f)$ Calculation of $x(t)$:

$$x(t) = Ae^{-\lambda t}\cos(\omega_a t + \varphi)$$

$$x(0) = A\cos\varphi = 2$$

$$x(100T_a) = Ae^{-\lambda 100 T_a}\cos(\omega_a 100T_a + \varphi)$$

$$x(100T_a) = Ae^{-100\delta}\cos(2\pi(100) + \varphi)$$

$$x(100T_a) = Ae^{-100x34}\cos\varphi$$

$$x(100T_a) = 2e^{-3400} \cong 0 = 0$$

Exercise 7 :

$$ln\left(\frac{x_1}{x_2}\right) = ln\left(\frac{18}{1}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \Longrightarrow \zeta = 0.4179$$

a) If the damping ratio ζ is doubled

$$ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi\,\zeta_{nouveau}}{\sqrt{1-\zeta_{nouveau}^2}} = \frac{2\pi\,(0{,}8358)}{\sqrt{1-(0{,}8358)^2}} \Longrightarrow \left(\frac{x_1}{x_2}\right) = 14265{,}362$$

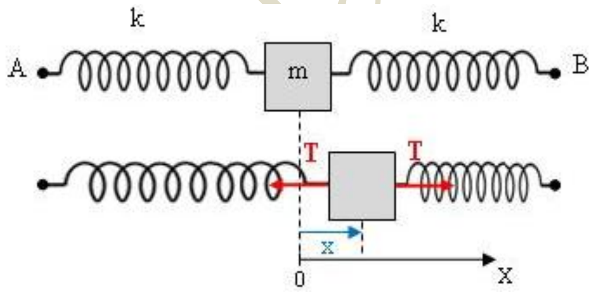
b) If the damping ratio ζ is reduced by half

$$ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi\,\zeta_{nouveau}}{\sqrt{1-\zeta_{nouveau}^2}} = \frac{2\pi\,(0{,}20895)}{\sqrt{1-(0{,}20895)^2}} \Longrightarrow \left(\frac{x_1}{x_2}\right) = 3.8286$$

Exercise 8 :

a) Points A and B are fixed.

We have the following system:



To find the natural frequency of the oscillations of mass m, we use:

$$\sum \vec{F} = m\vec{\gamma}$$

$$\Rightarrow m\ddot{x} = -kx - kx \qquad \Rightarrow m\ddot{x} + 2kx = 0$$

By dividing by m, we obtain : $\ddot{x} + \frac{2k}{m}x = 0$

It is a second-order differential equation of the form:

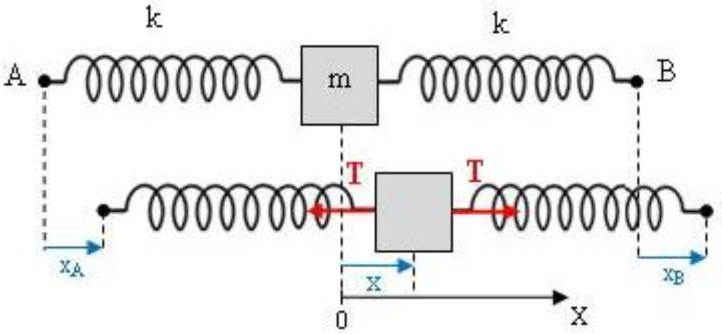
$$\ddot{x} + \omega_0^2 x = 0 \text{ with } \omega_0 = \sqrt{\frac{2k}{m}}$$

b) *Les points A et B sont mobiles* (*Points A and B are mobile).*

Les points A et B sont animés de vibrations dans la direction des X de la forme :

(Points A and B undergo vibrations in the X direction of the form:)

$$x_A(t) = -C\cos\omega_1 t \quad et \qquad x_B(t) = C\cos\omega_2 t$$



- **Determining $\boldsymbol{x(t)}$**

Along the (OX) axis, we have:

$$m\ddot{x} = -k(x - x_A) - k(x - x_B)$$

$$m\ddot{x} + 2kx = kx_A + kx_B$$

By dividing by m, we obtain: $\ddot{x} + \frac{2k}{m}x = \frac{k}{m}x_A + \frac{k}{m}x_B$ (1)

It is a second-order differential equation with a non-homogeneous term on the right-hand side

The motion equation $\boldsymbol{x(t)}$ of the mass can be obtained by superimposing these two motions. The general solution x(t) is of the form:

$$x_g(t) = x_h(t) + x_p(t)$$

Here $x_h(t)$ is the solution to the corresponding homogeneous equation

and $x_p(t)$ is a particular solution to the non-homogeneous equation

With :

$$x_h(t) = A\cos(\omega_0 t + \varphi) \qquad \omega_0 = \sqrt{\frac{2k}{m}}$$

The particular solution is written as:

$$x_p(t) = x_{p1}(t) + x_{p2}(t)$$

Knowing that:

$$x_{p1}(t) = B_1\cos(\omega_1 t + \varphi_1) \qquad et \qquad x_{p2}(t) = B_2\cos(\omega_2 t + \varphi_2)$$

$$\dot{x}_{p1}(t) = -\omega_1 B_1 \sin(\omega_1 t + \varphi_1) \qquad \ddot{x}_{p1}(t) = -\omega_1^2 B_1\cos(\omega_1 t + \varphi_1)$$

$$\dot{x}_{p2}(t) = -\omega_2 B_2 \sin(\omega_2 t + \varphi_2) \qquad \ddot{x}_{p2}(t) = -\omega_2^2 B_2\cos(\omega_2 t + \varphi_2)$$

We substitute into the differential equation (1), and we find:

$$-\omega_1^2 B_1 \cos(\omega_1 t+\varphi_1)+\frac{2k}{m}B_1\cos(\omega_1 t+\varphi_1)=-\frac{kC}{m}cos\omega_1 t+\frac{kC}{m}cos\omega_2 t$$

and, $-\omega_2^2 B_2 \cos(\omega_2 t+\varphi_2)+\frac{2k}{m}B_2\cos(\omega_2 t+\varphi_2)=-\frac{kC}{m}cos\omega_1 t+\frac{kC}{m}cos\omega_2 t$

So : $(-\omega_1^2+\frac{2k}{m})B_1\cos(\omega_1 t+\varphi_1)=-\frac{kC}{m}cos\omega_1 t+\frac{kC}{m}cos\omega_2 t$

and, $(-\omega_2^2+\frac{2k}{m})B_2\cos(\omega_2 t+\varphi_2)=-\frac{kC}{m}cos\omega_1 t+\frac{kC}{m}cos\omega_2 t$

By identification, we deduce that: $\varphi_1 = 0$ et $\varphi_2 = 0$,

And also : $\left(-\omega_1^2+\frac{2k}{m}\right)B_1=-\frac{kC}{m}\Rightarrow B_1=-\frac{\mathrm{kC}}{(2\mathrm{k}-\omega_1^2)}$

$$\left(-\omega_2^2+\frac{2k}{m}\right)B_2=\frac{kC}{m}\Rightarrow B_2=\frac{\mathrm{kC}}{(2\mathrm{k}-\omega_2^2)}$$

So :

$$x_{p1}(t)=-\frac{\mathrm{kC}}{(2\mathrm{k}-\omega_1^2)}\cos\ \omega_1 t$$

$$x_{p2}(t)=\frac{\mathrm{kC}}{(2\mathrm{k}-\omega_2^2)}\cos\omega_2 t$$

The general solution $x_g(t)$ is written as:

$$x_g(t)=A\cos\left(\sqrt{\frac{2k}{m}}\ t+\varphi\right)-\frac{\mathrm{kC}}{(2\mathrm{k}-\omega_1^2)}\cos\omega_1 t+\frac{\mathrm{kC}}{(2\mathrm{k}-\omega_2^2)}\cos\omega_2 t$$

- **Les valeurs des pulsations de résonance**

The values of ω_1 and ω_2 for which resonance will occur are those very close to

$$\omega_0=\sqrt{\frac{2k}{m}}.$$

If $\omega_1=\omega_2\Rightarrow x_{p1}(t)+x_{p2}(t)=0 \quad\Rightarrow\ x_h(t)=A\cos(\omega_0\, t+\varphi)$

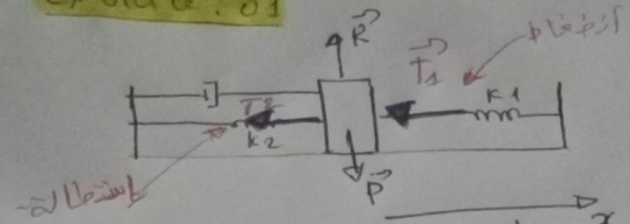
And the motion is harmonic with a pulsation ω_0.

Resonance Pulsations:

Resonance pulsations occur when the excitation frequency is equal to the natural frequency of the system (ω_0). In other words, when ω_1 or ω_2 is equal to ω_0, the system resonates. The values of ω_1 and ω_2 at which resonance occurs are therefore equal to ω_0.

For resonance, it is generally interesting to consider cases where ω_1 or ω_2 is equal to ω_0.

exercice : 01



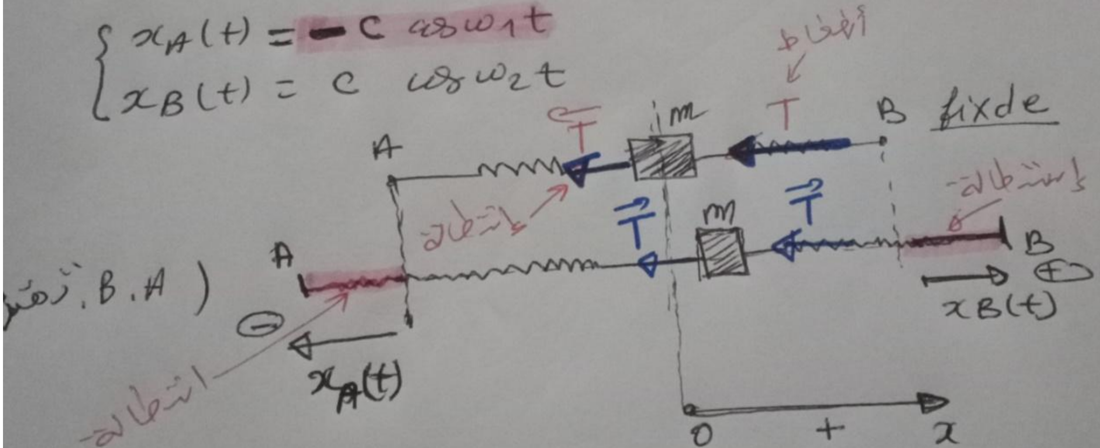
Ox : $\Sigma \vec{F} = m\vec{\gamma}$

$-K_1 x - K_2 x - F_{fr} = m\ddot{x}$

exercices : 08

b) les point A et B sont animé

$$\begin{cases} x_A(t) = - C \cos\omega_1 t \\ x_B(t) = C \cos\omega_2 t \end{cases}$$



$$m\ddot{x} = -K(x - x_B) - K(x - x_A)$$