Exam: Maths03

Duration: 02h:00

Exercise 01 (4.25 pts): Tell whether the following series converge or diverge:

- $\sum_{n=1}^{\infty}\left(1-\cos(n^{-\alpha})\right), \quad \alpha>0.$ 0,75
- $\sum_{n=0}^{\infty}\frac{n^2+1}{3n^2+2},$ 1
- $\sum_{n=1}^{\infty}\frac{2^n\left((n+1)!\right)^2}{(2n-1)!},$ 1,25
- $\sum_{n=0}^{\infty}\frac{2\sqrt{n}}{3^n}.$ 1,25

Exercise 02 (5.50 pts):

1. Study uniform convergence of the sequence of functions

$$f_n(x)=\frac{nx}{1+n^2x^2}, \quad x\in[0,1], \qquad n\geq 1.$$ 2,75

2. Let the power series

$$S(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{e^{n+2}}x^n, \quad x\in\mathbb{R}.$$ 2,75

   (a) Find the radius of convergence R.
   (b) Calculate the sum $S(x)$.

Exercise 03 (5.25 pts): Let $f(x)$ a 2π periodic function defined by

$$f(x)=\begin{cases}-x & \text{for} \quad -\pi\leq x\leq 0\\ x & \text{for} \quad 0<x\leq\pi\end{cases}$$

1. Expand the function $f(x)$ in a Fourier series, 4,25
2. Show that $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots$ 1

Exercise 04 (5 pts):

1. Calculate using polar coordinates the integral

$$\iint_D e^{-x^2-y^2}\,dx\,dy$$ 2,5

where

$$D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 9, \ x\geq 0, \ y\geq 0\}.$$

2. Without calculating, determine whether the following improper integrals converge or diverge:

   (a) $\int_2^{\infty}\frac{\ln x}{x\sqrt{x}}\,dx,$
   (b) $\int_0^1\frac{1-\cos(x)}{e^{x^2\sqrt{x}}-1}\,dx.$

2,5

Correction of the exam Math 03

Exercise 01 (4,25pt)

1. $\sum_{n\geq 1}(1-\cos(n^{-\alpha}))$, $\alpha>0$. We have $n^{-\alpha}=\frac{1}{n^{\alpha}}$, $\lim_{n\to\infty}\frac{1}{n^{\alpha}}=0$

$1-\cos\left(\frac{1}{n^{\alpha}}\right) \underset{0}{\sim} \frac{1}{2(n^{\alpha})^2}=\frac{1}{2n^{2\alpha}}$ (0,25 . $\frac{1}{2}\sum_{n\geq 1}\frac{1}{n^{2\alpha}}$ ↗ cv if $2\alpha>1$, $\alpha>\frac{1}{2}$ (0,25 ↘ div if $2\alpha\leq 1$, $\alpha\leq\frac{1}{2}$ (0,25

2. $\sum_{n\geq 1}\frac{2^n((n+1)!)^2}{(2n-1)!}$, We use ratio test (0,25, $\frac{u_{n+1}}{u_n}=\frac{(n+2)^2}{2n^2+n}$ (0,25

$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\frac{1}{2}<1$ (0,25, So the series cv. (0,5

3. $\sum_{n=0}^{\infty}\frac{n^2+1}{3n^2+2}$, $\lim_{n\to\infty}u_n=\frac{1}{3}\neq 0$ (0,5, the series div. (0,5

4. $\sum_{n=0}^{\infty}\frac{2^{\sqrt{n}}}{3^n}$, we use root test (0,25. $\sqrt[n]{u_n}=\frac{2^{\frac{1}{\sqrt{n}}}}{3}$ (0,25, $\lim_{n\to\infty}\sqrt[n]{u_n}=\frac{1}{3}<1$ (0,2

So the series cv. (0,5

Exercise 02 (5,5pt)

1. Pointwise cv: $\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{nx}{1+n^2x^2}=0$, $\forall x\in]0,1]$ (0,25

$f_n(0)=0$ (0,25, (f_n) cv. pointwise to $f(x)=0$, $\forall x\in[0,1]$ (0,25

2. Uniform cv. $\lim_{n\to\infty}\sup_{x\in[0,1]}|f_n(x)-f(x)|=0$? (0,25

$f_n'(x)=\frac{n-n^3x^2}{(1+n^2x^2)^2}$ (0,5, $f_n'(x)=0 \Rightarrow x=\frac{1}{n}$ (0,25

x	0		$\frac{1}{n}$		1
$f_n'(x)$		$+$	0	$-$	
$f_n(x)$		↗	$f_n(\frac{1}{n})$	↘	

(0,5

$\sup_{x\in[0,1]}|f_n(x)|=f_n\left(\frac{1}{n}\right)=\frac{1}{2}\neq 0$ (0,25

The convergence is not uniform (0,25

II) $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{e^{n+2}}x^n$, $|a_n|=\frac{1}{e^{n+2}}$, $\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{e^{n+3}}\cdot e^{n+2}$, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{e}$ (1

The radius of cv is $R=e$ (0,5

2. The sum: $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{e^{n+2}} x^n = \frac{-1}{e^2} \sum \left(\frac{-x}{e}\right)^n = \frac{-1}{e^2}\left(\frac{1}{1-\left(\frac{-x}{e}\right)}\right)$ 1

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{e^{n+2}} x^n = \frac{-1}{e^2+xe}$$ 0,25

Exercise 3 (5,25 pt)

1) $f(-x) = \begin{cases} x & -\pi \leq -x \leq 0 \\ -x & 0 < -x \leq \pi \end{cases} = \begin{cases} x & 0 \leq x \leq \pi \\ -x & -\pi \leq x < 0 \end{cases}$ 0,5

$= f(x) \Rightarrow f$ is even $\Rightarrow b_n = 0$ 0,5

$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$ 0,5

$a_n = \frac{2}{\pi}\int_0^{\pi} \underset{v}{x}\, \underset{u'}{\cos nx}\, dx = \frac{2}{\pi}\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{\sin nx}{n}dx$ 0,75

$= \frac{2}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi n^2}(\cos n\pi - 1)$ 0,5

$= \frac{2}{\pi n^2}\left((-1)^n - 1\right) = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{-4}{n^2\pi} & \text{if } n \text{ odd} \end{cases}$ 0,5

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ 0,5

$= \frac{\pi}{2} + \sum_{n=\text{odd}}^{\infty} \frac{-4}{n^2\pi}\cos nx = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=\text{odd}} \frac{\cos nx}{n^2}$ 0,5

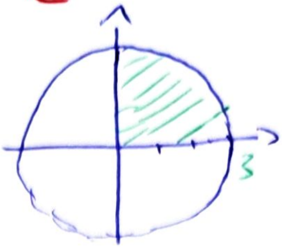
2) Deduce the sum, we replace $x=0$ in the fourier series expantion:

$f(0) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=\text{odd}}\frac{\cos 0}{n^2} \Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=\text{odd}}\frac{1}{n^2} \Rightarrow \frac{-\pi}{2}\cdot\frac{(-\pi)}{4} = \sum_{n\,\text{odd}}\frac{1}{n^2}$

So $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$ 1

Exercise 4

1- Polar coordinates : $\begin{cases} x = r\cos\theta & 0{,}25 \\ y = r\sin\theta & 0{,}25 \end{cases}$



$x^2+y^2 \leq 9 \Rightarrow 0 \leq r \leq 3$ 0,5

$x \geq 0,\ y \geq 0 \Rightarrow 0 \leq \theta \leq \frac{\pi}{2}$ 0,5

$f(r,\theta) = e^{-r^2}$ 0,25

$\Delta = [0,3] \times [0, \frac{\pi}{2}]$

$\iint_D f(x,y)\,dx\,dy = \iint_\Delta f(r,\theta)\, r\,dr\,d\theta$ 0,25

$= \int_0^{\pi/2}\int_0^3 r e^{-r^2}\,dr\,d\theta = \int_0^{\pi/2} d\theta \left(-\frac{1}{2}\int_0^3 -2re^{-r^2}\,dr\right)$ 0,25

$= \frac{\pi}{2}\left[-\frac{1}{2}e^{-r^2}\right]_0^3 = \frac{\pi}{4}\left(1-e^{-9}\right)$ 0,25

2- $I_1 = \int_2^\infty \frac{\ln x}{x\sqrt{x}}\,dx = \int_2^\infty \frac{1}{x^{3/2}(\ln x)^{-1}}\,dx$. Bertrand's integral 0,25 , $\alpha = \frac{3}{2} > 1$ 0,25

So the integral converges. 0,5

$I_2 = \int_0^1 \frac{1-\cos x}{e^{x^2\sqrt{x}}-1}\,dx$, $1-\cos x \underset{0}{\sim} \frac{x^2}{2}$ 0,25 ; $e^{x^2\sqrt{x}} - 1 \underset{0}{\sim} x^2\sqrt{x}$ 0,25

$\frac{1-\cos x}{e^{x^2\sqrt{x}}-1} \underset{0}{\sim} \frac{x^2}{2x^{5/2}} = \frac{x^{-1/2}}{2} = \frac{1}{2x^{1/2}}$ 0,25

$\frac{1}{2}\int_0^1 \frac{1}{x^{1/2}}\,dx$ cv. by p test $\alpha = \frac{1}{2} < 1$ 0,5

By equivalent theorem $\int_0^1 \frac{1-\cos x}{e^{x^2\sqrt{x}}-1}\,dx$ converges 0,25