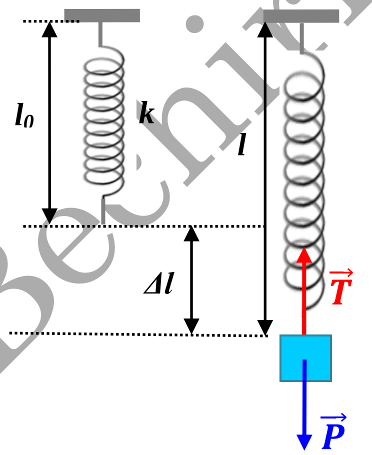
University of Badji Mokhtar-Annaba **2025-2026**
Faculty of Engineering Sciences
Department of Science and Technology,
2nd Year
<u>Pr. L. BECHIRI</u> **Tutorial 1 Solutions**
Module: Physics 3

<u>Exercise 1:</u>

Determination of the angular frequency of the system:

According to the fundamental relation of dynamics

(The equation of motion from Newton's law is)



$$\sum \vec{F} = m\vec{\gamma}$$

The system at rest, thus in equilibrium, we have:

$$\sum \vec{F} = \vec{0} \quad \Longrightarrow \quad \vec{P} + \vec{T} = \vec{0}$$

(tension $\vec{T}$ or restoring force ($\vec{F}_r$)

$\Longrightarrow \quad mg - k\Delta l = 0, \quad \Delta l = Spring\ elongation$

$$mg = k\Delta l \quad \Longrightarrow \quad \boldsymbol{k = \frac{mg}{\Delta l}}$$

The Lagrangian : $L = T - V$

kinetic energy of the system : $T = \frac{1}{2}m\dot{x}^2$

Potential energy of the spring : $V = \frac{1}{2}kx^2$

$$\Longrightarrow \quad L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

The Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right)\right] = \frac{d}{dt}(m\dot{x}) = \boldsymbol{m\ddot{x}}$$

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{2}kx^2\right) = \boldsymbol{-kx}$$

$$\Longrightarrow \quad m\ddot{x} + kx = 0$$

$$\Longrightarrow \quad \ddot{x} + \frac{k}{m}x = 0$$

$$\begin{cases} \ddot{x} + \frac{k}{m}x = 0 \\ \ddot{x} + \omega_0^2 x = 0 \end{cases} \Rightarrow \omega_0^2 = \frac{k}{m} \Rightarrow \omega_0 = \sqrt{\frac{k}{m}}$$

$$the\ angular\ frequency\ \omega_0 = \sqrt{\frac{\frac{mg}{\Delta l}}{m}} = \sqrt{\frac{g}{\Delta l}} = \sqrt{\frac{9.81}{7{,}787.10^{-3}}}$$

$$\Rightarrow \quad \omega_0 = 35{,}49\ rad/s$$

Exercise 2:

For a harmonic oscillator:

$$x(t) = A\cos(\omega_0 t \pm \varphi)$$

With A as the amplitude and the angular frequency ω_0

$$\omega_0^2 = \frac{k}{m}$$

$$E_{\text{mechanical}} = T + V = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2 = \frac{1}{2}kA^2 = \frac{1}{2}m\omega_0^2 A^2$$

$$A = \sqrt{\frac{2E_m}{m\omega_0^2}}$$

$$A = 0{,}224m$$

Exercise 3:

a) The mass m of the solid.

$$\omega = 8rad/s \Rightarrow \omega^2 = \frac{k}{m}, \Rightarrow m = \frac{k}{\omega^2} = \frac{48}{8^2} = 0.75kg$$

b) The amplitude of the motion..

$$E_m = E_c + E_p = \frac{1}{2}kx_m^2 = 0.24$$

$$x_m^2 = \frac{2}{k}0.24 = 0.01, x_m = 0.1m = 10cm$$

c) The maximum velocity of the oscillator.

$$x = x_m \sin(8t - \pi)$$

$$\dot{x} = 8\text{x}x_m\, cos\,(8t - \pi)$$

$$v_m = 8\text{x}10 = 80cm/s$$

d) The elongation of the oscillator for which the potential energy is equal to half of the kinetic energy.

$$E_m = E_c + E_p == 0.24$$

$$E_m = E_c + \frac{1}{2}E_c == 0.24$$

$$E_c + \frac{1}{2}E_c = 0.24, \quad 3E_c = 2\text{x}0.24 = 0.48$$

$$E_c = \frac{1}{3}0.48 = 0.16$$

$$E_c = \frac{1}{2}mv^2 = \frac{1}{2}m(8\text{x}x_m cos(8t-\pi))^2 = \frac{1}{2}0.75\text{x}64\text{x}x_m^2(cos(8t-\pi))^2 = 0.16$$

$$x_m = \mp 8{,}16 \text{ cm}$$

e) The components of velocity and acceleration at this point.

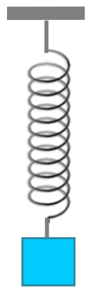
$$v_x = \dot{x} = 8\text{x}x_m\, cos\,(8t-\pi)$$

$$v_x = \mp 8\text{x}0.0816 = \mp 0.65 m/s$$

$$\gamma_x = \ddot{x} = 64\text{x}x_m\, sin\,(8t-\pi)$$

$$\gamma_x = \mp 64\text{x}0.0816 = \mp 5.22 m/s^2$$

Exercise 4:



$\omega_0^2 = \frac{k}{m} = 4$ à $t = 0,\ x_0 = 2cm\,, \dot{x}_0 = 8cm/s$

The resulting motion is : $x(t) = Acos(\omega_0 t + \varphi)$

such that $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{4} = 2\ rad/s$

à $t = 0$ $on\ a$: $x(0) = Acos\varphi = 2 \ldots\ldots\ldots\ldots\ldots (\mathbf{1})$

$\dot{x}(t) = -A\omega sin(\omega_0 t + \varphi)$

$$\dot{x}(0) = -A\omega_0 sin\varphi = 8 \ldots\ldots\ldots\ldots (\mathbf{2})$$

$$\frac{(\mathbf{2})}{(\mathbf{1})} = \frac{-A\omega_0 sin\varphi}{Acos\varphi} = -\omega_0 tan\varphi = \frac{-8}{2} = 4 \Rightarrow tan\varphi = \frac{-4}{\omega_0} = \frac{-4}{2} = -2$$

$$\Rightarrow \quad \varphi = -63{,}43° = -1{,}10\ rad$$

$$Acos\varphi = 2 \Rightarrow A = \frac{2}{cos\varphi} = 4{,}47\ cm$$

$$x(t) = 4{,}47cos(2t - 1{,}10) \quad (cm)$$

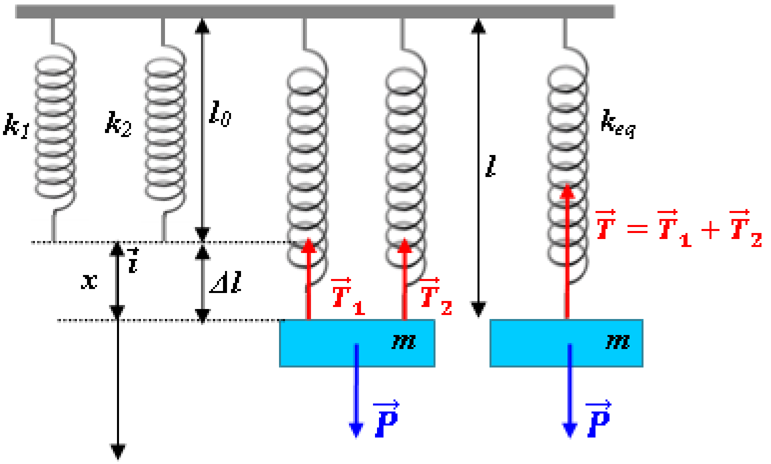
The maximum acceleration: $\ddot{x}(t) = -A{\omega_0}^2 cos(\omega_0 t + \varphi)$

Alors $\ddot{x}_{max} = |-A{\omega_0}^2| = 4{,}47 \times 2^2 = 17{,}88\ cms^{-2}$

Exercise 5:

The equivalent stiffness constants k_{eq}

a) Springs in parallel :



Springs 1 and 2 are subjected to tension or restoring forces $\vec{T}_1$ $(\vec{F}_{r1})$ and $\vec{T}_2$ $(\vec{F}_{r2})$ respectively, defined by:

$$\vec{T}_1 = k_1 \Delta l \vec{i} \quad \text{et} \quad \vec{T}_2 = k_2 \Delta l \vec{i}$$

$$\Delta l = l - l_0 = x$$

At equilibrium, we have:

$$\sum \vec{F} = \vec{0} \Longrightarrow \vec{P} + \vec{T}_1 + \vec{T}_2$$

After projection onto the axis of motion, we have:

$$P - T_1 - T_2 = 0$$

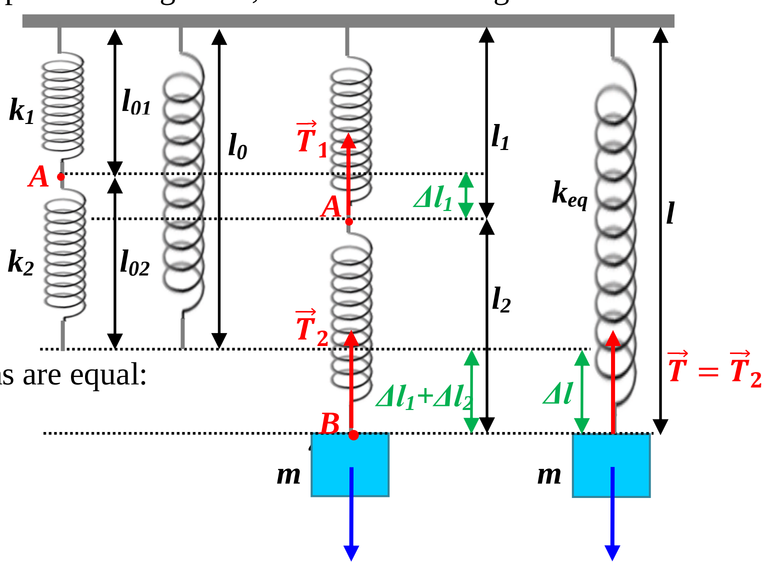
$$mg - k_1 x - k_2 x = 0 \Longrightarrow mg = k_1 x + k_2 x$$

$$\Longrightarrow mg = (k_1 + k_2)x = k_{eq} x$$

With $\boldsymbol{k_{eq}} = \boldsymbol{k_1} + \boldsymbol{k_2}$ for the parallel system.

b) Springs in series :

Let l_{01}, l_{02} and l_0 be the unstressed lengths of springs 1, 2, and the equivalent spring, and Δ Δl_1, Δl_2 and Δl their respective elongations, as shown in the figure below



At point A, the tensions are equal:

$$T_1 = T_2 \implies k_1 \Delta l_1 = k_2 \Delta l_2 \; \dots \dots \dots \dots \dots (1)$$

Where: $\Delta l_1 = l_1 - l_{01}$

$$\Delta l_2 = l_2 - l_{02}$$

At point B, at equilibrium, we have:

$$\begin{cases} mg = k_2 \Delta l_2 \\ \Delta l = \Delta l_1 + \Delta l_2 \end{cases}$$

According to equation (1), we find:

$$\Delta l_1 = \frac{k_2}{k_1} \Delta l_2$$

$$\implies \Delta l = \frac{k_2}{k_1} \Delta l_2 + \Delta l_2 = \left(\frac{k_2}{k_1} + 1\right) \Delta l_2$$

$$\implies \Delta l_2 = \left(\frac{k_1}{k_1 + k_2}\right) \Delta l$$

By substituting this expression into the equilibrium equation, we obtain:

$$mg = k_2 \left(\frac{k_1}{k_1 + k_2}\right) \Delta l = \left(\frac{k_1 k_2}{k_1 + k_2}\right) \Delta l$$

$$\implies \qquad k_{eq} = \frac{k_1 k_2}{k_1 + k_2} \qquad \text{for the series system}$$

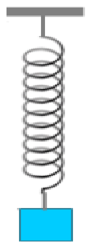
$$\text{or} \quad \frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

Analogy with equivalent capacitances C_{eq} of capacitors: it is known that for capacitors in series $\frac{1}{C_{eq}} = \sum_i \frac{1}{C_i}$ and for capacitors in parallel :$C_{eq} = \sum_i C_i$

The calculation of the equivalent capacitance in an electrical oscillator is analogous to the calculation of the equivalent stiffness in a mechanical oscillator.

Exercise 6:

Determination of k_2 as a function of k_1 : $(k_1, m_1 = m)$



$$L = T - V$$

$$L = \frac{1}{2} m_1 \dot{x}^2 - \frac{1}{2} k_1 x^2$$

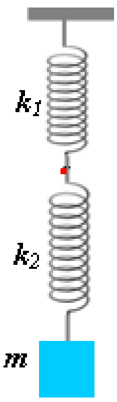
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2} m_1 \dot{x}^2\right)\right] = \frac{d}{dt}(m_1 \dot{x}) = m_1 \ddot{\boldsymbol{x}}$$

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{2} k_1 x^2\right) = -k_1 \boldsymbol{x}$$

$$\Longrightarrow m_1\ddot{x} + k_1 x = 0$$

$$\Longrightarrow \quad \ddot{x} + \frac{k_1}{m_1}x = 0$$



$\Longrightarrow \omega_1^2 = \frac{k_1}{m_1} \Longrightarrow \omega_1 = \sqrt{\frac{k_1}{m_1}}$ The natural frequency

-A second spring is added in series with the first spring

$$\left(k_{eq}, \quad m_1\right) \longrightarrow \omega_2 = \frac{1}{2}\omega_1$$

According to the previous answer, we find the following differential equation:

$$\ddot{x} + \frac{k_{eq}}{m_1}x = 0$$

with $\omega_2^2 = \frac{k_{eq}}{m_1} \Longrightarrow \omega_2 = \sqrt{\frac{k_{eq}}{m_1}}$

The equivalent spring constant k_{eq} in the series system is:

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2}$$

$$\Longrightarrow \omega_2 = \sqrt{\frac{\frac{k_1 k_2}{k_1 + k_2}}{m_1}}$$

We have : $\omega_2 = \frac{1}{2}\omega_1 \Longrightarrow \omega_2^2 = \frac{1}{4}\omega_1^2$

$$\Longrightarrow \frac{k_1 k_2}{k_1 + k_2} \cdot \frac{1}{m_1} = \frac{1}{4}\frac{k_1}{m_1}$$

$$\Longrightarrow k_1 + k_2 = 4k_2$$

$$\Longrightarrow k_1 = 3k_2$$

$$\Longrightarrow k_2 = \frac{1}{3}k_3$$

<u>Exercise 7 :</u>

The Lagrangian of the system**:**

$$L = T - V$$

The kinetic energy of the system is given by**:**

$$T = T_M + T_m = \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2$$

$$\vec{v}_M = \frac{d\overrightarrow{OA}}{dt} = \dot{\boldsymbol{x}}\vec{\boldsymbol{\imath}}$$

$$\Longrightarrow T_M = \frac{1}{2}Mv_M^2 = \frac{1}{2}M\dot{\boldsymbol{x}}^{\mathbf{2}}$$

$$\vec{v}_m = \frac{d\overrightarrow{OB}}{dt} = \frac{d}{dt}\left(\overrightarrow{OA} + \overrightarrow{AB}\right)$$

$$\overrightarrow{OA}\begin{pmatrix} x \\ 0 \end{pmatrix}, \qquad \overrightarrow{AB}\begin{pmatrix} lsin\theta \\ -lcos\theta \end{pmatrix}$$

$$\Longrightarrow \vec{v}_m = \frac{d}{dt}(x\vec{\imath} + lsin\theta\vec{\imath} - lcos\theta\vec{\jmath}) =$$

$$= \dot{x}\vec{\imath} + l\dot{\theta}cos\theta\vec{\imath} + l\dot{\theta}sin\theta\vec{\jmath} = \left(\dot{x} + l\dot{\theta}cos\theta\right)\vec{\imath} + l\dot{\theta}sin\theta\vec{\jmath}$$

$$v_m^2 = x^2 + y^2$$

$$v_m^2 = \left(\dot{x} + l\dot{\theta}cos\theta\right)^2 + \left(l\dot{\theta}sin\theta\right)^2 = \dot{x}^2 + l^2\dot{\theta}^2cos^2\theta + 2\dot{x}l\dot{\theta}cos\theta + l^2\dot{\theta}^2sin^2\theta$$

We have : $$cos^2\theta + sin^2\theta = 1$$

$$\Longrightarrow v_m^2 = \dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\dot{\theta}cos\theta$$

$$T_m = \frac{1}{2}mv_m^2 = \frac{1}{2}m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\dot{\theta}cos\theta\right)$$

$$\Longrightarrow T_m = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + m\dot{x}l\dot{\theta}cos\theta$$

$$T = T_M + T_m$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + m\dot{x}l\dot{\theta}cos\theta$$

$$T = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + m\dot{x}l\dot{\theta}cos\theta$$

The potential energy of the system is**:**

$$V = V_M + V_m$$

$$V_M = \frac{1}{2}kx^2$$

$$V_m = mgh$$

we have : $$cos\theta = \frac{L-h}{L} \Longrightarrow L - h = Lcos\theta$$

$$\Longrightarrow h = L - Lcos\theta = L(1 - cos\theta)$$

$$\Longrightarrow V_m = mgL(1 - cos\theta)$$

$$V = V_M + V_m = \frac{1}{2}kx^2 + mgL(1 - cos\theta)$$

$$L = T - V$$

$$\Longrightarrow L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + m\dot{x}l\dot{\theta}cos\theta - \frac{1}{2}kx^2 - mgL(1 - cos\theta)$$

Note:

This system has two independent variables x and θ ⟹ a system with two degrees of freedom. To derive the differential equation, it will be necessary to apply the Lagrange equations for the two variables, that is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0 \; et \; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = 0$$

<u>Exercise 8:</u>

Determination of the angular frequency of the oscillatory motion:

$$L = T - V$$

The kinetic energy :

$$T = T_{translation} + T_{rotation}$$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2$$

With J as the moment of inertia,

We have : $x = R\theta \Longrightarrow \dot{x} = R\dot{\theta} \Longrightarrow \dot{\theta} = \frac{\dot{x}}{R}$

$$\Longrightarrow T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\frac{\dot{x}^2}{R^2}$$

$$\Longrightarrow T = \frac{1}{2}\left(m + \frac{J}{R^2}\right)\dot{x}^2$$

The potential energy :

$$V = \frac{1}{2}kx^2$$

So: $L = \frac{1}{2}\left(m + \frac{J}{R^2}\right)\dot{x}^2 - \frac{1}{2}kx^2$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0$$

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}\left(m + \frac{J}{R^2}\right)\dot{x}^2\right)\right] = \frac{d}{dt}\left(m + \frac{J}{R^2}\right)\dot{x} = \left(m + \frac{J}{R^2}\right)\ddot{x}$$

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{2}kx^2\right) = -kx$$

$$\left(m + \frac{J}{R^2}\right)\ddot{x} + kx = 0$$

$$\Longrightarrow \ddot{x} + \frac{k}{\left(m + \frac{J}{R^2}\right)}x = 0$$

$$\Longrightarrow \omega = \sqrt{\frac{k}{\left(m + \frac{J}{R^2}\right)}}$$

For a cylinder, the moment of inertia is given by $J = \frac{1}{2} mR^2$

So : $\omega = \sqrt{\frac{2k}{3m}}$ *The natural angular frequency of the system*

Exercise 9:

Determination of the equation of motion:

$$L = T - V$$

The kinetic energy :

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J\dot{\theta}^2$$

We have : $x = R\theta \Longrightarrow \dot{x} = R\dot{\theta} \Longrightarrow \dot{\theta} = \frac{\dot{x}}{R}$

So

$$T = \frac{1}{2}(m + \frac{J}{R^2})\dot{x}^2$$

-The potential energy :

$$V = \frac{1}{2} kx^2$$

-Lagrangian :

$$L = \frac{1}{2}\left(m + \frac{J}{R^2}\right)\dot{x}^2 - \frac{1}{2} kx^2$$

So, the Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0$$

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}(\frac{1}{2}\left(m + \frac{J}{R^2}\right)\dot{x}^2)\right] = \frac{\partial}{\partial t}\left[\left(m + \frac{J}{R^2}\right)\dot{x}\right] = \left(m + \frac{J}{R^2}\right)\ddot{x}$$

$$\frac{\partial}{\partial x}\left(-\frac{1}{2} kx^2\right) = -kx$$

Therefore, the differential equation of motion is written as:

$$\left(m + \frac{J}{R^2}\right)\ddot{x} + kx = 0$$

$$\Longrightarrow x + \frac{k}{(m + \frac{J}{R^2})} x = 0$$

With a natural angular frequency: $\omega_0 = \sqrt{\frac{k}{\frac{J}{R^2} + m}}$

The solution to the differential equation is of the form:

$$x(t) = Acos(\omega_0 t + \varphi)$$

We have : $at \quad t = 0s \longrightarrow x(0) = 4cm \;\; et \;\; \dot{x}(0) = 0$

$$x(at\ t = 0) = Acos\varphi = 4cm \qquad (\mathbf{1})$$

$$\dot{x}(t) = -A\omega_0 \sin(\omega_0 t + \varphi)$$

$$\dot{x}(0) = -A\omega_0 sin\varphi = 0$$

$$-A\omega_0 \neq 0 \Longrightarrow sin\varphi = 0$$

$$\Longrightarrow \varphi = 0$$

$$(\mathbf{1}) \Longrightarrow Acos0 = 4cm$$

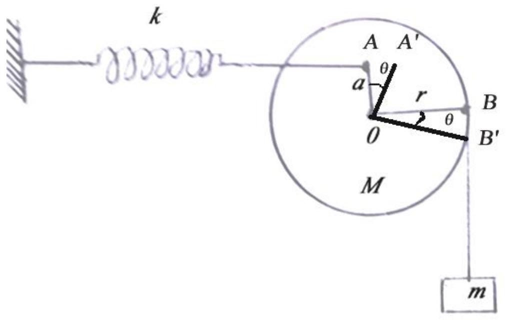
$$\Longrightarrow A = 4cm$$

Therefore, we can write the equation of motion as:

$$x(t) = Acos(\omega_0 t) = 4\cos\left(\sqrt{\frac{k}{\frac{J}{R^2} + m}}\, t\right) \ (cm)$$

<u>Exercise 10 :</u>

<u>Both the Lagrangian and energy conservation methods are equally valid and must be accepted as correct solutions</u>



First method: Lagrangian Mechanics Analysis

Generalized Coordinate: Angular displacement θ of the disc.

The kinetic energy of the system **(T):**

- **Rotational kinetic energy of the disc:**

$$T_{\text{disc}} = \frac{1}{2} I_{\text{disc}} \dot{\theta}^2$$

- **Translational kinetic energy of the mass m:**

  Mass m translation at B', $x = r\theta$, so its velocity is $\dot{x} = r\dot{\theta}$. Thus:

$$T_{\text{m}} = \frac{1}{2} m\,(r\dot{\theta})^2 = \frac{1}{2} m r^2 \dot{\theta}^2$$

- **Total kinetic energy:**

$$KE = T = T_{\text{disc}} + T_{\text{m}} = \frac{1}{2} I_{\text{disc}} \dot{\theta}^2 + \frac{1}{2} m\,(r\dot{\theta})^2 = \frac{1}{2} I_{\text{disc}} \dot{\theta}^2 + \frac{1}{2} m r^2 \dot{\theta}^2$$

Potential Energy (V):

- **Elastic potential energy of the spring:**

  Spring elongation at A', $x = a\theta$, so its potential energy is:

  $PE = V = V_s = \frac{1}{2} k\,(a\theta)^2 = \frac{1}{2} k a^2 \theta^2$

- The Lagrangian is defined as:

$$L = T - V$$

  Substitute T and V

$$L = T - V == \frac{1}{2} I_{\text{disc}} \dot{\theta}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - \frac{1}{2} k a^2 \theta^2$$

The Euler-Lagrange equation is:

- $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$$\frac{\partial L}{\partial \dot{\theta}}: \ \frac{\partial L}{\partial \dot{\theta}} = (I_{\text{disk}} + m r^2)\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right): \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = (I_{\text{disc}} + m r^2)\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta}: \ \frac{\partial L}{\partial \theta} = -k a^2 \theta$$

Equation of Motion:

$$(I_{\text{disc}} + m r^2)\ddot{\theta} - (-\,k a^2 \theta) = 0$$

The equation of motion is: $(I_{\text{disc}} + m r^2)\ddot{\theta} + k a^2 \theta = 0$

Simple Harmonic Motion:

$$\ddot{\theta} + \left(\frac{ka^2}{I_{\text{disc}}+mr^2}\right)\theta = 0 \Longrightarrow \ddot{\theta} + \left(\frac{ka^2}{\frac{1}{2}Mr^2+mr^2}\right)\theta = 0 \Longrightarrow \ddot{\theta} + \omega_n^2\theta = 0$$

where: $\omega_n = \sqrt{\frac{ka^2}{I_{\text{disc}}+mr^2}}, \quad \omega_n = \sqrt{\frac{ka^2}{\frac{1}{2}Mr^2+mr^2}}$ (Natural frequency)

Second method: Energy Conservation

The kinetic energy of the system **(T):**

- **Rotational kinetic energy of the disc:**

$$T_{\text{disc}} = \frac{1}{2} I_{\text{disc}}\dot{\theta}^2$$

- **Translational kinetic energy of the mass m:**

  Mass m translation $x = r\theta$, so its velocity is $\dot{x} = r\dot{\theta}$. Thus:

$$T_{\text{m}} = \frac{1}{2} m\,(r\dot{\theta})^2 = \frac{1}{2} mr^2\dot{\theta}^2$$

- **Total kinetic energy:**

$$T = T_{\text{disc}} + T_{\text{m}} = \frac{1}{2} I_{\text{disc}}\dot{\theta}^2 + \frac{1}{2} m\,(r\dot{\theta})^2 = \frac{1}{2} I_{\text{disc}}\dot{\theta}^2 + \frac{1}{2} mr^2\dot{\theta}^2 = \frac{1}{2}(I_{\text{disc}} + mr^2)\dot{\theta}^2$$

Potential Energy (V):

- **Elastic potential energy of the spring:**

  Spring elongation $x = a\theta$, so its potential energy is:

  $PE = V = V_s = \frac{1}{2} k\,(a\theta)^2 = \frac{1}{2} ka^2\theta^2$

  Energy Conservation:

  Total mechanical energy is constant: T + V = constant

  Substitute T and V

$$E = T + V == \frac{1}{2} I_{\text{disc}}\dot{\theta}^2 + \frac{1}{2} mr^2\dot{\theta}^2 + \frac{1}{2} ka^2\theta^2 = \frac{1}{2}(I_{\text{disc}} + mr^2)\dot{\theta}^2 + \frac{1}{2} ka^2\theta^2$$

Differentiate the energy equation with respect to time.

- $\frac{dE}{dt} = \frac{d(T+V)}{dt} = 0$
- (Applying the Chain Rule to find the derivative of a composition of functions
- $\frac{d\dot{\theta}^2}{dt} = \frac{d\dot{\theta}^2}{d\theta}\frac{d\theta}{dt} = 2\ddot{\theta}\dot{\theta}$)

  $(I_{\text{disc}} + mr^2)\ddot{\theta}\dot{\theta} + ka^2\theta\dot{\theta} = 0$

For non-zero $\dot{\theta}$ ($\frac{d\theta}{dt} \neq 0$):

Equation of Motion:

$$(I_{\text{disc}} + mr^2)\ddot{\theta} + ka^2\theta = 0$$

Simple Harmonic Motion:

$$\ddot{\theta}+\left(\frac{ka^2}{I_{\text{disc}}+mr^2}\right)\theta=0 \Longrightarrow \ddot{\theta}+\left(\frac{ka^2}{\frac{1}{2}Mr^2+mr^2}\right)\theta=0 \Longrightarrow \ddot{\theta}+\omega_n^2\theta=0$$

where: $\omega_n=\sqrt{\frac{ka^2}{I_{\text{disc}}+mr^2}}$, $\omega_n=\sqrt{\frac{ka^2}{\frac{1}{2}Mr^2+mr^2}}$ (Natural frequency)

Exercise 11 :

$$L=T-V$$

The kinetic energy :

$$T=\frac{1}{2}m\dot{x}^2$$

-The potential energy :

$$V=\frac{1}{2}k_1x^2+\frac{1}{2}k_2y^2$$

For small oscillations (small angles θ), we have

$$\theta\simeq sin\theta=\frac{x}{b}=\frac{y}{a}$$

So : $y=\frac{a}{b}x$

$$V=\frac{1}{2}k_1x^2+\frac{1}{2}k_2\frac{a^2}{b^2}x^2 \Longrightarrow V=\frac{1}{2}\left(k_1+\frac{a^2}{b^2}k_2\right)x^2$$

$$L=\frac{1}{2}m\dot{x}^2-\frac{1}{2}\left(k_1+\frac{a^2}{b^2}k_2\right)x^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\left(\frac{\partial L}{\partial x}\right)=0$$

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right)\right]=\frac{d}{dt}(m\dot{x})=m\ddot{x}$$

$$\left[\frac{\partial}{\partial x}\left(-\frac{1}{2}\left(k_1+\frac{a^2}{b^2}k_2\right)x^2\right)\right]=-\left(k_1+\frac{a^2}{b^2}k_2\right)x$$

$$m\ddot{x}+\left(k_1+\frac{a^2}{b^2}k_2\right)x=0$$

$$\Longrightarrow \ddot{x}+\frac{\left(k_1+\frac{a^2}{b^2}k_2\right)}{m}x=0$$

With : $\omega=\sqrt{\frac{\left(k_1+\frac{a^2}{b^2}k_2\right)}{m}}$ natural angular frequency

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