Series 02: Sequences and series of functions

Exercice 01 : *Study pointwise and uniform convergence,in the domain I for the sequence functions $(f_n(x))_{n\geq 0}$:*

- $f_n(x) = xe^{-nx}, \quad I = [0, +\infty[,$
- $f_n(x) = \dfrac{1-nx^2}{1+nx^2}, \quad I = \mathbb{R}, \text{ and } I = [a, +\infty[, \ a > 0,$
- $f_n(x) = cos\left(\dfrac{1+nx}{2+n}\right), \quad I = [-a, a], \ a > 0.$

Exercice 02 : *Let the sequence functions $(f_n(x))_{n\geq 0}$:*

$$f_n(x) = \frac{2^n x}{1 + n2^n x^2} \quad , x \in [0, 1].$$

- Calculate $\lim_{n\to\infty} \int_0^1 f_n(x)dx$ and $\int_0^1 \lim_{n\to\infty} f_n(x)dx$,
- Deduce that the convergence of $(f_n(x))_{n\geq 0}$ is not uniform.

Exercice 03 : *Find the domain of convergence of the following series:*

- $\sum_{n=0}^{\infty} (-1)^{n+1} \dfrac{x^n}{\sqrt{n}},$
- $\sum_{n=0}^{\infty} n^{(e^{-|x|}-2)}$
- $\sum_{n=0}^{\infty} \left(\dfrac{ne+1}{n+2}\right)^n (e^{-x})^n$

Exercice 04 :*Show that the series $\sum_{n=0}^{\infty} f_n(x)$ converges normally in $\mathbb{R}^+$:*

- $f_n(x) = \sqrt{n}xe^{-n^2x},$
- $f_n(x) = \dfrac{cos(nx)}{n^2 + x}.$

Deduce the uniform and the absolute convergence of the series $\sum_{n=0}^{\infty} f_n(x)$.