BADJI MOKHTAR - ANNABA UNIVERSITY **2025-2026**
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Vibration and Waves
Solutions of Final Examination

Solution 1: (4points)

1-Equivalent Spring Constant Calculation:
A parallel combination of springs with stiffness k and $3k$:

$$k_{parallel} = k + 3k = 4k \qquad \textbf{(0.25)}$$

This combination is in series with a spring of stiffness $4k$. The equivalent stiffness k_{eq} is given by

$$\frac{1}{k_{eq}} = \frac{1}{k_{parallal}} + \frac{1}{2k} = \frac{1}{4k} + \frac{1}{4k} = \frac{2}{4k} \qquad \textbf{(0.25)}$$

The equivalent spring constant : $k_{eq} = 2k = 10\ N/m$ **(0.25)**
Therefore, the entire spring system can be replaced by a single spring with stiffness $2k$.
2- Equation of motion using Newton's second law. Thus, the restoring force exerted by the spring system is given by: $F_r = -k_{eq}x$
Applying Newton's second Law:

$$m\ddot{x} = -k_{eq}x \Longrightarrow m\ddot{x} + k_{eq}x = 0 \Longrightarrow m\ddot{x} + 2kx = 0 \qquad \textbf{(0.5)}$$

This is the equation of motion for the system.
3-The angular natural frequency ω_n and the period T_0 in terms of m and k

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{2k}{m}}, \qquad T_0 = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{2k}} \qquad \textbf{(0.5+0.5)}$$

4- Amplitude–phase form and numerical evaluation

$$x(t) = Acos(\omega_n t + \varphi)$$

$$x(0) = Acos(\omega_n 0 + \varphi) = Acos\varphi = 0.02$$

$$Acos\varphi = 0.02 \qquad (1) \qquad \textbf{(0.25)}$$

$$v(t) = \dot{x}(t) = -A\omega_n \sin(\omega_n t + \varphi) \qquad \textbf{(0.25)}$$

$$v(0) = \dot{x}(0) = -A\omega_n \sin(\omega_n 0 + \varphi) = -A\omega_n sin\varphi = 0.01$$

$$Asin\varphi = -\frac{0.01}{\omega_n} \qquad (2) \qquad \textbf{(0.25)}$$

$$m = 2kg \text{ and } k = 5N/m \qquad \omega_n = \sqrt{\frac{2k}{m}} = \sqrt{\frac{2x5}{2}} = 2.24 \text{ rad/s}$$

$$A = \sqrt{(1)^2 + (2)^2}$$

$$A = \sqrt{0.02^2 + \left(-\frac{0.01}{\omega_n}\right)^2} = 0.0205 \qquad \textbf{(0.25)}$$

$$\frac{(2)}{(1)} = tan\varphi = -\frac{0.01}{0.02\text{x}\omega_n} \Longrightarrow \varphi = \tan^{-1}\left(-\frac{0.01}{0.02\text{x}\omega_n}\right) = -0.22\ radians \qquad \textbf{(0.25)}$$

$$x(t) = 2.05cos(2.24t - 0.22)cm \qquad \textbf{(0.5)}$$

Solution 2: (10points)

1- This system is a single-degree-of-freedom, viscously damped, forced oscillator. Using Newton's second law, $m\vec{\gamma} = \Sigma\vec{F_i} = \vec{F_r} + \vec{F_f} + \overrightarrow{F_{ext}}$

The equation of motion for a driven oscillator is, in general,

$$m\ddot{x} + c\dot{x} + m\omega_0^2 x = F_0 cos\omega_{ex} t \quad or \quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m} cos\omega_{ex} t \quad \textbf{(0.5)}$$

where ω_0 is the natural angular frequency and ω_{ex} is the driving angular frequency. The damping constant c has units of $N\, s/m$ when velocity $\dot{x}$ is in ms^{-1} and force is in N.

Given: $m = 0.6\, kg$, $\omega_0 = \omega_n = 20$ rad/ s, $c = 4.8\, N\, s/m$, $F_0 = 2.4$N and $\omega_{ex} = 12\, rad/s$, and spring constant: $k = m\omega_n^2 = 6x20^2 == 240$N/m

The equation of motion is simply

$$0.6\ddot{x} + 4.8\dot{x} + 240x = 2.4cos12t \quad \textbf{(0.5)}$$

or, equivalently, after dividing by the mass $m = 0.6\, kg$:

$$\ddot{x} + 8\dot{x} + 400x = 4cos12t \;\; \textit{(in standard form)} \quad \textbf{(0.5)}$$

2. Determining the Characteristic Parameters

-Natural Period T_0

$$\omega_n^2 = \frac{k}{m} = 400\text{rad/s} \quad \Rightarrow \omega_n = 20\text{rad/s} \Rightarrow \quad T_0 = \frac{2\pi}{\omega_n} = \frac{2\pi}{20} = 0.314\text{s} \quad \textbf{(0.5 +0.5)}$$

-Damping Coefficient λ

$$2\lambda = \frac{c}{m} = 8 \Rightarrow \lambda = 4 \quad \textbf{(0.5)}$$

-Excitation Angular Frequency (Driving angular frequency) ω_{ex}:

$$\omega_{ex} = 12rad/s \quad \textbf{(0.5)}$$

3. Show transient is oscillatory damped motion, find pseudo-frequency ω_a,

Transient solution : $\ddot{x} + 8\dot{x} + 400x = 0$ **(0.5)**

The characteristic equation for the homogeneous part is:

$r^2 + 2\lambda r + \omega_0^2 = 0 \Rightarrow r^2 + 8r + 400 = 0$ with roots: $r = -4 \pm j19.5959$ **(0.5 +0.5)**

Discriminant: $\Delta' = \lambda^2 - \omega_0^2 \quad (\Delta' = b'^2 - ac) \Rightarrow \Delta' = 16 - 400 = -384 < 0$ system is underdamped. **(0.5)**

$$\text{(or Damping ratio: } \boldsymbol{\zeta} = \frac{\boldsymbol{\lambda}}{\boldsymbol{\omega_n}} = \frac{\mathbf{4}}{\mathbf{20}} = \mathbf{0.2 < 1})$$

Since the roots are complex with a negative real part, the transient solution is a damped oscillatory motion. The damped angular frequency is:

$$\omega_a = \sqrt{(\omega_0^2 \ - \lambda^2)} = \omega_0\sqrt{1-\zeta^2} = 20\sqrt{1-0.2^2} = 19.5959 \text{ rad/s} \qquad \textbf{(0.5)}$$

Homogeneous solution: $x_h(t) = Ae^{-\lambda t}cos(\omega_a t - \theta)$

$$x_h(t) = Ae^{-4t}cos(19.5959\, t - \theta) \qquad \textbf{(0.5)}$$

4- Steady-state solution, amplitude, and phase lag

The steady-state displacement is of the form : $x_p(t) = X_0 cos(\omega_{ex} t - \varphi)$, where: X_0 the amplitude:

$$X_0 = |\overline{X}| = \frac{F_0/m}{\sqrt{\left(\omega_0^2 - \omega_{ex}^2\right)^2 + (c\omega_{ex}/m)^2}} \approx 0.0146\text{m}, \qquad \textbf{(0.5 +0.5)}$$

while φ the phase lag

$$tan\varphi = \left(\frac{c\omega_{ex}/m}{\omega_0^2 - \omega_{ex}^2}\right) = 0.375 \quad \Longrightarrow \varphi \approx 0.3588\text{rad}\,,\ 20.556^\circ) \qquad \textbf{(0.5 +0.5)}$$

Final steady-state:

$$x_p(t) = X_0 cos(\omega_{ex} t - \varphi) = 0.0146\text{x}cos(12t - 0.3588)m \qquad \textbf{(0.5)}$$

5-General solution:

The general solution is the sum of the transient solution $x_h(t)$ and the steady-state solution $x_p(t)$:

$$x_g(t) = x_h(t) + x_p(t) \qquad \textbf{(0.5)}$$

$$x_g(t) = Ae^{-4t}cos(19.5959\, t - \theta) + 0.01463\text{x}cos(12t - 0.3588)m \qquad \textbf{(0.5)}$$

where A and θ are determined by the initial conditions.

Solution 3: (6points)

1- Differential Equations of Motion

Using Newton's second law for each mass, the equations of motion are:

$$m\ddot{x}_1 = -k(x_1 - x_2) \quad \textbf{(0.25)}$$

$$m\ddot{x}_2 = -k(x_2 - x_1) - k(x_2 - x_3) \quad \textbf{(0.25)}$$

$$m\ddot{x}_3 = -k(x_3 - x_2) \quad \textbf{(0.25)}$$

$$\text{Left block}: \quad m\ddot{x}_1 + kx_1 - kx_2 = 0 \quad \textbf{(0.25)}$$

$$\text{Middle block}: \quad m\ddot{x}_2 + 2kx_2 - kx_1 - kx_3 = 0 \quad \textbf{(0.25)}$$

$$\text{Right block}: \quad m\ddot{x}_3 + kx_3 - kx_2 = 0 \quad \textbf{(0.25)}$$

2- Natural Angular Frequencies

Assuming harmonic solutions of the form:

$$x_1(t) = A\cos(\omega t + \varphi) \quad \textbf{(0.25)}$$

$$x_2(t) = B\cos(\omega t + \varphi) \quad \textbf{(0.25)}$$

$$x_3(t) = C\cos(\omega t + \varphi) \quad \textbf{(0.25)}$$

Substituting these into the equations of motion (1) and simplifying, we get a system of linear equations:

$$\begin{cases} (k - m\omega^2)A - kB = 0 \\ -kA + (2k - m\omega^2)B - kC = 0 \\ -kB + (k - m\omega^2)C = 0 \end{cases} \quad \textbf{(0.25+0.25+0.25)}$$

This can be written in matrix form as:

$$\mathfrak{D} = \begin{vmatrix} (k - m\omega^2) & -k & \\ -k & (2k - m\omega^2) & -k \\ & -k & (k - m\omega^2) \end{vmatrix} \begin{vmatrix} A \\ B \\ C \end{vmatrix} = 0 \quad \textbf{(0.5)}$$

The system is homogeneous. There are non-trivial solutions only if the determinant $\mathfrak{D}$ of the coefficient matrix is zero. The coefficients are polynomials in ω. The permissible values are the solutions of $\mathfrak{D} = 0$. **(0.25)**

The natural angular frequencies of the system are the roots of the equation:

$$(k - m\omega^2)\big((2k - m\omega^2)(k - m\omega^2) - k^2\big) + k(-k(k - m\omega^2) = 0 \quad \textbf{(0.25)}$$

$$\omega^2(k - m\omega^2)(-3km + m^2\omega^2) = 0 \quad \textbf{(0.5)}$$

Thus, the natural angular frequencies are:

$$\Rightarrow \omega_1 = 0$$

$$\Rightarrow \omega_2 = \sqrt{\frac{k}{m}} \ rad/s \quad \textbf{(0.25+0.25+0.25)}$$

$$and \qquad \Longrightarrow \omega_3 = \sqrt{\frac{3k}{m}} \quad rad/s$$

3- General Solutions for Displacements

The general motion is a superposition of the three normal modes, The homogeneous solutions are then:

$x_1(t) = A_1 \cos(\omega_1 t + \varphi_1) + A_2 \cos(\omega_2 t + \varphi_2) + A_3 \cos(\omega_3 t + \varphi_3)$ **(0.25)**

$x_2(t) = B_1 \cos(\omega_1 t + \varphi_1) + B_2 \cos(\omega_2 t + \varphi_2) + B_3 \cos(\omega_3 t + \varphi_3)$ **(0.25)**

$x_3(t) = C_1 \cos(\omega_1 t + \varphi_1) + C_2 \cos(\omega_2 t + \varphi_2) + C_3 \cos(\omega_3 t + \varphi_3)$ **(0.25)**

Where A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3 φ_1, φ_2 and φ_3 are constants determined by the initial conditions of the system