Series 1

Exercise 1 :

1. Show that : $\forall x, y \in \mathbb{R}^2, x^2 + y^2 = 0 \Longleftrightarrow x = y = 0$.
2. Show that : $x \neq y \Longrightarrow (x+1)(y-1) + (x-1)(y+1)$.
3. Show that : n is prime $\Longrightarrow n = 2$ or n is odd.

Exercise 2 :

1. Show by the absurd that : $\forall x \in \mathbb{R} : x \notin \mathbb{Q} \Longrightarrow 1 + x \notin \mathbb{Q}$.
2. Show by recurrence : $\forall n \geq 0, \quad 6^n + 9$ is a multiple of 5.

Exercise 3 :

Consider the following four assertions :
(a) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \quad x + y > 0$. (b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \quad x + y > 0$.
(c) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \quad x + y > 0$. (d) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \quad y^2 > x$.

1. Are the assertions a, b, c and d true or false ?
2. Give their negation.

Exercise 4 :

In $\mathbb{C}$, we define the relation $\mathcal{R}$ by :

$$z\mathcal{R}z' \Longleftrightarrow |\, z \,|=|\, z' \,| \,.$$

Show that $\mathcal{R}$ is an equivalence relation.

Exercise 5 :

Let the application $f : \mathbb{R} \longmapsto \mathbb{R}$ defined by $f(x) = x^2 + 1$. Consider the sets $A = [-3, 2], \quad B = [0, 4]$.

1. Compare the sets $f(A \cap B)$ and $f(A) \cap f(B)$.
2. What condition must f satisfy for $f(A \cap B) = f(A) \cap f(B)$.

Exercise 6 :

1. Let $f : \mathbb{R} \longmapsto \mathbb{R}$ be the application defined by :

$$f(x) = \begin{cases} 1 & x < 0 \\ 1 + x & x \geqslant 0. \end{cases}$$

   Determine the following sets : $f(\mathbb{R}), f^{-1}(0), f^{-1}(1), f^{-1}(-1), f^{-1}(]1, 2])$.

2. Let $g : \mathbb{R} \setminus \{\frac{1}{2}\} \to \mathbb{R}^*$ be the application such that :

$$g(x) = \frac{9}{2x-1}.$$

   Show that g is a bijection. Determine its reciprocal application.

Additional exercises

Exercise 1 : Show that : [n is odd] $\Longleftrightarrow$ [n^2 is odd]

Exercise 2 :

1. Let be the application $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ and $A = [-1; 4]$. Determine :
   (a) The direct image of A by f.
   (b) The reciprocal image of A by f.
2. What is the direct image of sets : $\mathbb{R}$, $[0, 2\pi]$, $[0, \frac{\pi}{2}]$, and the reciprocal image of sets$[0, 1]$, $[3, 4]$, $[1, 2]$ by application $\sin(x)$.