Correction of the exam

Exercise 01 (4,5 pts)

1. $\sum_{n\geq 0} u_n$, $u_n = \left(\frac{n+1}{n+3}\right)^{n^2} e^{-2n}$, we use Root test (Cauchy test) (0,25)

$$\lim_{n\to+\infty} \sqrt[n]{u_n} = \lim_{n\to+\infty} \sqrt[n]{\left(\frac{n+1}{n+3}\right)^{n^2} e^{-2n}} \quad (0,25)$$

$$= \lim_{n\to+\infty} \left(\frac{n+1}{n+3}\right)^{n} e^{-2} = \lim_{n\to\infty} \left(\frac{n+3-2}{n+3}\right)^{n} e^{-2} \quad (0,25)$$

$$= \lim_{n\to\infty} \left(1-\frac{2}{n+3}\right)^{n} e^{-2} \quad (0,25)$$

$$= e^{-2} . e^{-2} = e^{-4} < 1 \quad (0,25)$$

So $\sum_{n\geq 0} u_n$ Converges (0,5)

2. $\sum_{n\geq 0} u_n$, $u_n = \frac{n!}{3^{n/2}}$, we use Ratio test (d'Alembert test) (0,25)

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to+\infty} \frac{(n+1)!}{3^{\frac{(n+1)}{2}}} . \frac{3^{n/2}}{n!} \quad (0,25)$$

$$= \lim_{n\to+\infty} \frac{(n+1)}{3^{1/2}} = +\infty \quad (0,25)$$

So $\sum_{n\geq 0} u_n$ diverges (0,5)

3. $\sum_{n\geq 0} \frac{(-1)^n}{5+n^2}$, $|u_n| = \frac{1}{5+n^2} \leq \frac{1}{n^2}$ (0,25)

$\sum \frac{1}{n^2}$ p-series (Rieman series) $\alpha = 2 > 1$ (0,25)

So $\sum \frac{1}{n^2}$ converges (0,25)

$\sum_{n\geq 0} u_n$ is absolutely convergent (0,25)

So $\sum_{n\geq 0} u_n$ converges (0,25)

Or : We use alternating series test (Leibniz) (0,25)

* $v_n = \frac{1}{5+n^2} > 0$ (0,25)
* $\lim_{n\to\infty} v_n = 0$ (0,25)
* v_n is decreasing (0,25)

So $\sum_{n\geq 0} u_n$ Converges (0,5)

Exercise 02 (3,45 pts)

1.1. Pointwise convergence

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{e^{nx}} = 0, \forall x \in \mathbb{R}^+$ (0,25)

So (f_n) conv. pointwise to $f(x)=0, \forall x\in\mathbb{R}^+$ (0,25)

1.2 Uniform conv. "$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^+} |f_n(x)-0|$" (0,25)

$f_n'(x) = \frac{n}{e^{nx}}(1-nx)$ (0,25)

$f_n'(x) = 0 \Rightarrow x = \frac{1}{n}$ (0,25)

x	0	$\frac{1}{n}$	$+\infty$
$f_n'(x)$	$+$	0	$-$
$f_n(x)$	$\nearrow$	$f_n(1/n)$	$\searrow$

(0,25) (0,25)

$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^+} |f_n(x)| = f_n\left(\frac{1}{n}\right) = \frac{1}{e} \neq 0$ (0,5)

So (f_n) does not conv. uniformly to $f(x)=0$ (0,5)

2. $\sum_{n\geq 0} \frac{\sin^2 x}{n^2+1}, x\in\mathbb{R}$

$f_n(x) = \frac{\sin^2 x}{n^2+1} \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}$ (0,5)

$\sum \frac{1}{n^2}$ is convergent (0,25)

So $\sum_{n\geq 0} f_n(x)$ conv. normally (0,25)

Exercise 3 (5,75 pts)

1. $S(x) = \sum_{n\geq 2} \frac{(-1)^{n-2}}{n(n-1)} x^n$, $a_n = \frac{(-1)^{n-2}}{n(n-1)}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n(n-1)}{(n+1)n}\right| = 1$ (0,5)

So $R = 1$ (0,5)

2. $S'(x) = \sum_{n\geq 2} \frac{(-1)^{n-2}\, n x^{n-1}}{n(n-1)}, \forall x \in]-1,1[$ (0,25)

$S'(x) = \sum_{n\geq 2} \frac{(-1)^{n-2}}{(n-1)} x^{n-1}$ (0,25)

$S''(x) = \sum_{n\geq 2} (-1)^{n-2} x^{n-2}$ (0,5)

$= \frac{1}{1+x}$ (0,75)

3. By integration

$S'(x) = \int_0^x S''(t)\,dt = \int_0^x \frac{1}{1+t}\,dt$ (0,25)

$= \left[\ln(1+t)\right]_0^x = \ln(1+x)$ (0,5)

$S(x) = \int_0^x S'(t)\,dt = \int_0^x \ln(1+t)\,dt$ (0,25)

(0,25)

$u' = 1 \Rightarrow u = t$

$v = \ln(1+t) \Rightarrow v' = \frac{1}{1+t}$ (0,25)

$S(x) = \left[t\ln(1+t)\right]_0^x - \int_0^x \frac{t}{1+t}\,dt$ (0,25)

$= x\ln(1+x) - \int_0^x \frac{t+1-1}{1+t}\,dt$ (0,25)

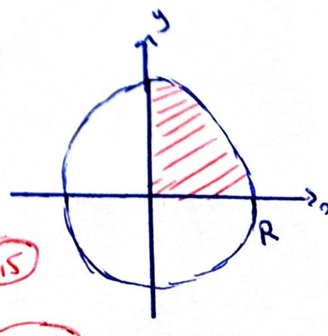
$= x\ln(1+x) - \int_0^x dt + \int_0^x \frac{dt}{1+t}$ (0,25)

$= x\ln(1+x) - \left[t\right]_0^x + \left[\ln(1+t)\right]_0^x$ (0,25)

$S(x) = (1+x)\ln(1+x) - x$ (0,5)

Exercise 4 (6 pts)

1. Polar coordinates



$x = r\cos\theta$ (0,25)

$y = r\sin\theta$ (0,25)

$x^2+y^2 \leq R^2 \Rightarrow 0\leq r\leq R$ (0,5)

$x\geq 0, y\geq 0 \Rightarrow 0\leq\theta\leq\frac{\pi}{2}$ (0,5)

$f(r,\theta) = e^{-r^2}$ (0,25)

$\Delta = [0, R] \times [0, \frac{\pi}{2}]$

$\iint_D f(x,y)\,dx\,dy = \iint_\Delta f(r,\theta)\, r\, dr\, d\theta$ (0,25)

$= \int_0^{\frac{\pi}{2}} \int_0^R r e^{-r^2} dr\, d\theta$ (0,25)

$= \int_0^{\frac{\pi}{2}} d\theta \times \left(-\frac{1}{2} \int_0^R -2 r e^{-r^2} dr\right)$ (0,25)

$= \frac{\pi}{2} \cdot \left[-\frac{1}{2} e^{-r^2}\right]_0^R$ (0,25)

$= \frac{\pi}{4}\left(1 - e^{-R^2}\right)$ (0,5)

2. Improper integrals

$I_1 = \int_1^{+\infty} \frac{\cos \pi x}{x^2} dx$

$0 < \left|\frac{\cos \pi x}{x^2}\right| \leq \frac{1}{x^2}$ (0,25)

$\int_1^{\infty} \frac{1}{x^2} dx$ conv. $\left(\begin{matrix}\text{p-test} \\ (\text{Riemann})\end{matrix}^{\alpha = 2 > 1}\right)$ (0,5)

So $\int_1^{\infty} \frac{\cos \pi x}{x^2} dx$ is absolutely cv. (0,25)

Therefore it is convergent (0,25)

$I_2 = \int_0^1 \frac{\ln\left(1 + \sqrt[3]{x}\right)}{e^{\sin x} - 1} dx$

$\ln\left(1 + \sqrt[3]{x}\right) \underset{0}{\sim} \sqrt[3]{x}$ (0,25)

$e^{\sin x} - 1 \underset{0}{\sim} \sin x \underset{0}{\sim} x$ (0,5)

$\frac{\ln\left(1 + \sqrt[3]{x}\right)}{e^{\sin x} - 1} \underset{0}{\sim} \frac{x^{1/3}}{x} = \frac{1}{x^{2/3}}$ (0,25)

$\int_0^1 \frac{1}{x^{2/3}} dx$ Conv. (0,25)

So $\int_0^1 \frac{\ln\left(1 + \sqrt[3]{x}\right)}{e^{\sin x} - 1} dx$ converges (0,25)